8.4. Unitary Operators

Inner product preserving

- V, W inner product spaces over F in R or C.
- T:V -> W.
- T preserves inner products if (Ta|Tb) = (a|b) for all a, b in V.
- An isomorphism of V to W is a vector space isomorphism T:V -> W preserving inner products.
- ||Ta|| = ||a||.

- Theorem 10. V, W f.d. inner product spaces. dim V = dim W. TFAE.
  - (i) T preserve inner product
  - (ii) T is an inner product space isomorphism.
  - (iii) T carries every orthonormal basis of V to one of W.
  - (iv) T carries some orthonormal basis of V to one of W.
- Proof. (iv)->(i). Use $(Ta_i, Ta_j) = (a_i, a_j)$. Then $a=x_1a_1+\ldots+x_na_n$, $b=y_1a_1+\ldots+y_na_n$,
  Prove $(Ta|Tb)=(a|b)$.

Corollary. V, W f.d. inner product spaces over F. Then V, W is isomorphic iff dim V = dim W.

- Proof: Take any basis $\{a_1, \ldots, a_n\}$ of V and a basis $\{b_1, \ldots, b_n\}$ of W. Let T:V -> W be so that $Ta_i = b_i$. Then by Theorem 10, T is an isomorphism.
- Theorem 11. V, W, inner product spaces over F. Then T perserves ips iff $||Ta|| = ||a||$ for all a in V.

- Definition: A *unitary operator* of an inner product space V is an isomorphism V-> V.
- The product of two unitary operators is unitary.
- The inverse of a unitary operator exists and is unitary. (by definition, it exists.)
- U is unitary iff for an orthonormal basis $\{a_1, \ldots, a_n\}$, we have an orthonormal basis $\{Ua_1, \ldots, Ua_n\}$

Theorem 12. Let U be a linear operator of an ips V. Then U is unitary iff U^* exists and $U^*U=I$, $UU^*=I$.

- Proof: $(Ua|b) = (Ua|UU^{-1}b)=(a|U^{-1}b)$ for all a, b in V.
- Conversely, assume that U^* exists and $U^*U=I=UU^*$. Then $U^{-1}=U^*$.
- $(Ua|Ub) = (a|U^*Ub)=(a|b)$. U is a unitary operator.
- **Definition:** A complex matrix A is unitary if $A^*A=I$.

- A real or complex matrix A is *orthogonal* if $A^tA = I$.
- A real matrix is unitary iff it is orthogonal.
- A complex unitary matrix is orthogonal iff it is real. (<- easy, -> $A^t = A^{-1} = A^*$)
- Theorem 14. Given invertible nxn matrix B, there exists a unique lower-triangular matrix M with positive diagonals so that MB is unitary.

- Proof: Basis $\{b_1, .., b_n\}$, rows of B.
  - Gram-Schmidt orthogonalization gives us

$$a_k = b_k - \sum_{j<k} \frac{(b_k \mid a_j)}{\| a_j \|^2} a_j \text{ gives us } a_k = b_k - \sum_{j<k} C_{kj} b_j$$

  - Let U be a unitary matrix with rows $a_i / || a_i ||$
  -

Let M be given by
Lower-triangular
Use $r_i(AB)=r_i(A)B$
$=r_{i1}(A)b_1+..+r_{in}(A)b_n$
Then U=MB

$$M_{kj} = \begin{cases} -\frac{1}{\| a_k \|} C_{kj}, j < k \\ \frac{1}{\| a_k \|}, j = k \\ 0, j > k \end{cases}$$

- Uniqueness: M_1, M_2 so that M_iB is unitary.
- $M_1B\,(M_2B)^{-1} = M_1\,(M_2)^{-1}$ is unitary.
- Lower triangular with positive entries also.
- This implies this has to be I.
- $T^+(n)$:= {lower triangular matrices with positive diagonals}
- This is a group. (i.e., product, inverse are also in $T^+(n)$, use row operations obtaining inverses to prove this.)

- Corollary. B in GL(n). There exists unique N in $T^+(n)$ and U in U(n) so that B = NU.
- Proof: B=NU for N unique by Theorem 14. Since $U = N^{-1} B$, U is unique also.

- B is *unitarily equivalent* to A if $B = P^{-1} A P$ for a unitary matrix P.
- B is *orthogonally equivalent* to A if $B = P^{-1} A P$ for an orthogonal matrix P.

8.5. Normal operators

- V f.d. inner product space.
- T is *normal* if T*T = TT*.
- Self-adjoint operators are normal. (generalization of self-adjoint property)
- We aim to show these are diagonalizable.
- Theorem 15. V inner product space. T is a self-adjoint operator. Then each eigenvalues are real. For distinct eigenvalues the eigenvectors are orthogonal.

- Proof: Ta = ca. Then c(a|a) = (ca|a)=(Ta|a) = (a,Ta)=(a|ca)=c^{-}(a|a). Thus c=c^{-}.
- Tb=db. Then c(a|b)=(Ta|b)=(a|Tb)= d^{-}(a|b)=d(a|b). Since c ≠b, (a|b)=0.
- Theorem 16. V f.d. ips. Every self-adjoint operators has a nonzero eigenvector.
- Proof.   det(xI –A) has a root. A-cI is singular.

For infinite dim cases, a self-adjoint operator may not have any nonzero eigenvector. See Example 29.

- Theorem 17. V f.d.ips. T operator. If W is a T-inv subspace, then $W^{\perp}$ is T* invariant.
- Proof: a in W -> Ta in W. Let b in $W^{\perp}$. (Ta|b)= 0 for all a in W. Thus (a|T*b) =0 for all a in W. Hence, T*b is in $W^{\perp}$.
- Theorem 18. V f.d.ips. T self-adjoint operator. Then there is an orthonormal basis of eigenvectors of T.
- Proof. Start from one a. W = <a>. Take $W^{\perp}$ invariant under T. And T is still self-adjoint there. By induction we are done.

- Corollary, nxn hermitian matrix A. There exists a unitary matrix P s.t. $P^{-1}AP$ is diagonal.
- nxn orthogonal matrix A. There exists an orthogonal matrix P s.t. $P^{-1}AP$ is diagonal.
- Theorem 19. V f.d.ips. T normal operator. Then a is an eigenvector for T with value c iff a is an eigenvector for T* with value c^-.
- Proof: $||Ua||^2=(Ua|Ua) = (a|U^*Ua)=(a|UU^*a)=(U^*|U^*a)=||U^*a||^2$.
- $U=T-cI$ is normal. $U^*=T^*-c^-I$. $||T-cI(a)|| = ||T^*-c^-I(a)||$.

- Definition: A complex nxn matrix A is called normal iff AA* = A*A.
- Theorem 20. V f.d.ips. B orthonormal basis. Suppose that the matrix A of T is upper triangular. Then T is normal if and only if A is a diagonal matrix.
- Proof: (<-) B is orthonormal.
  If A is diagonalizable, A*A = AA*. Hence, T*T=TT*.
- (->) T normal. $Ta_1 = A_{11}\, a_1$ since A is upper triangular. Thus, $T^*a_1 = A_{11}^{-}\, a_1$ by Theorem 19. Thus $A_{1j}=0$ for all $j > 1$.

- $A_{12} = 0$. Thus, $Ta_2 = A_{22}\, a_2$. Thus, $T^*a_2 = A_{22}^{-}\, a_2$
- Induction A is diagonal.
- Theorem 21. V. f.d.ips. T a linear operator on V. Then there exists an orthonormal basis for V where the matrix of T is upper triangular.
- Proof. Take an eigenvector a of T*. T*a=ca.
  Let W_1 be the orthogonal complement of a.
- W_1 is invariant under T by Th. 17. dim W_1 =n-1. By induction assumption, we obtain an orthonormal basis $a_1, a_2,.., a_{n-1}$. Add $a=a_n$
- Then T is upper triangular. (Ta_i is a sum of $a_1,\ldots,a_i$)

- Corollary. For any complex nxn matrix A , there is a unitary matrix U s.t. $U^{-1}AU$ is upper triangular.
- Theorem 22. V f.d.i.p.s. T is a normal operator. Then V has an orthonormal basis of eigenvectors on T.
- Corollary. **Every normal matrix A has a unitary matrix P such that $P^{-1}AP$ is a diagonal matrix.**