# Tropical computational experiments with *compactifications* of the *deformation spaces of convex real projective structures* on 2-orbifolds and surfaces.

S. Choi

Department of Mathematical Science
KAIST, Daejeon, South Korea
mathsci.kaist.ac.kr/ schoi (the copy of the lecture)

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joint work with Daniele Alessandrini.

# Abstract

Abstract: We present some compactification methods using the traces of the closed curves on surfaces. The basic methods are Daniele Alessandrini's work and the trace computations on the deformation spaces. We present some experimental evidences.

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- We need only consider very good orbifolds of form $S/G$ for a surface $S$ and a finite group $G$ acting on it.

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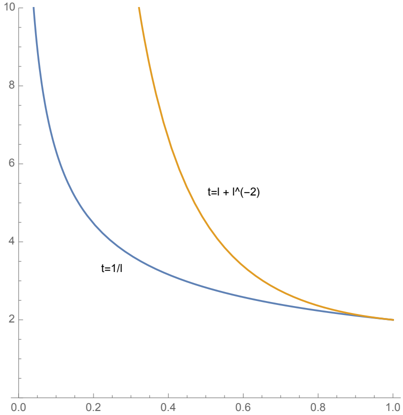
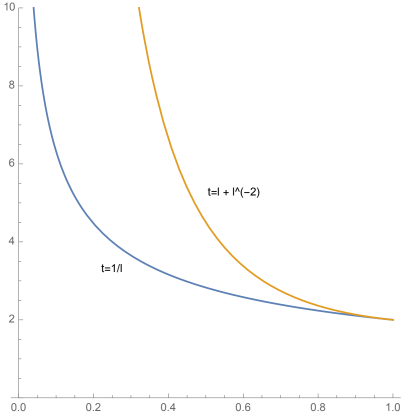
- $A^n, n = 1, 2, ..$ acts with an attractive fixed point and a repelling fixed point and a saddle type fixed point. The three lines connecting any two are $A$-invariant.
- The conjugation invariants of a positive hyperbolic element $A$ are eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with
$$0 < \lambda_1 < \lambda_2 < \lambda_3, \lambda_1\lambda_2\lambda_3 = 1.$$




- The Goldman invariants of *A* are given by $\lambda, \tau$ where $\lambda = \lambda_1, \tau = \lambda_2 + \lambda_3$.

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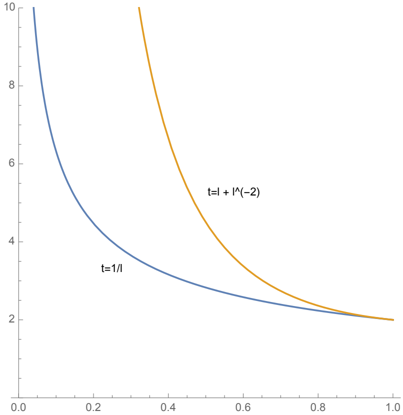


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- $A$ is positive hyperbolic iff $(\lambda(A), \tau(A)) \in D^o$.
- The arcs correspond to quasi-hyperbolic automorphisms.
- The point $(1, 2)$ to a parabolic one. (See Goldman [JDG1990].)

# Deformation spaces of nice convex real projective structures

- Given a closed or open orbifold $S$ of finite type, a convex real projective structure is given by a diffeomorphism $f : S \to \Omega/\Gamma$ for a convex domain $\Omega$ in $\mathbb{R}P^n$ and $\Gamma$ a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$.

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- The ends are required to be either be a cusp, has a ideal principal geodesic boundary for hyperbolic case, and ideal geodesic boundary for quasi-hyperbolic case. These are nice structures.
- $\mathrm{CDef}(S) = \{f : S \to \Omega/\Gamma\}/\sim$ where $f \sim g$ for $f : S \to \Omega/\Gamma$ and $g : S \to \Omega'/\Gamma'$ if there exists a projective diffeomorphism $k$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & \Omega/\Gamma \arrow[d, "k"] \\ & \Omega'/\Gamma' \end{tikzcd} \tag{1}$$

so that $k \circ f$ is isotopic to $g$.

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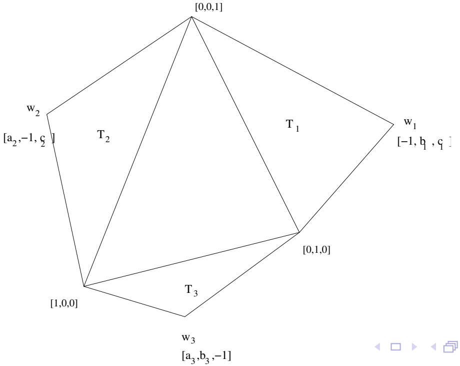
- Let $H_1$ be defined by $(\lambda(h(g_i)), \tau(h(g_i))) \in D(g_i)$ for $g_i$ representing ideal boundary components of $S$.

$$\mathrm{hol} : \mathrm{CDef}(S) \to H_1/\mathrm{SL}(3, \mathbb{R})$$

is a homeomorphism to a union of components of $H_1$. (Alessandrini-Choi, Marquis)

# Computing CDef(Pair of pants)

- Let $P$ be a pair of pants with $A, B, C$ the generator of $\pi_1(P)$ satisfying $CBA = I$.



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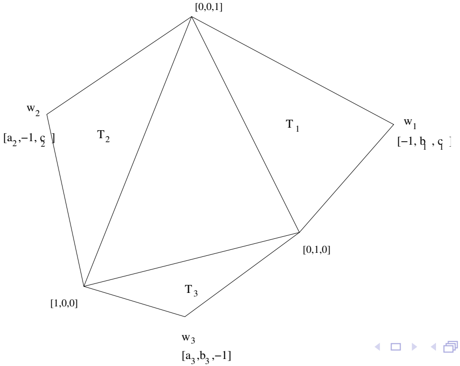
- Then $\mathrm{CDef}(P)$ is parameterized by
$$(s, t, \lambda(A), \tau(A), \lambda(B), \tau(B), \lambda(C), \tau(C))$$
where $s, t \in \mathbb{R}^2_+$ and the rest are in $D_1 \times D_2 \times D_3$ for $D_1 = D(A), D_2 = D(B), D_3 = D(C)$ above.



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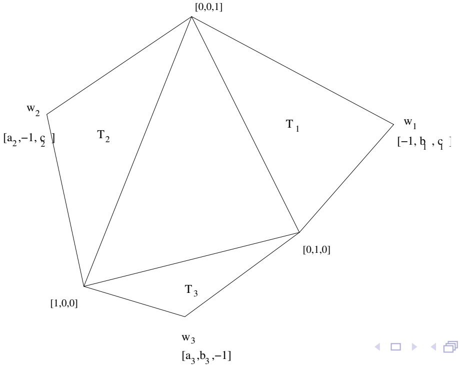
  where $s, t \in \mathbb{R}^2_+$ and the rest are in $D_1 \times D_2 \times D_3$ for $D_1 = D(A), D_2 = D(B), D_3 = D(C)$ above.
- $\mathrm{CDef}(P) = \mathbb{R}^2_+ \times D_1 \times D_2 \times D_3$. (following L. Marquis.)



# The matrices

First, the matrices must assume these forms:

$$\begin{aligned} A &:= \begin{bmatrix} \alpha_1 & \alpha_1 a_2 + \gamma_1 c_2 a_3 & \gamma_1 a_3 \\ 0 & -\beta_1 + \gamma_1 b_3 c_2 & \gamma_1 b_3 \\ 0 & -\gamma_1 c_2 & -\gamma_1 \end{bmatrix} \\ B &:= \begin{bmatrix} -\alpha_2 & 0 & -\alpha_2 a_3 \\ -\alpha_2 b1 & \beta_2 & \beta_2 b_3 + \alpha_2 a_3 b_1 \\ \alpha_2 c_1 & 0 & -\gamma_2 + \alpha_2 a_3 c_1 \end{bmatrix} \\ C &:= \begin{bmatrix} -\alpha_3 + \beta_3 a_2 b_1 & \beta_3 a_2 & 0 \\ -\beta_3 b_1 & -\beta_3 & 0 \\ \gamma_3 c_1 + \beta_3 b_1 c_2 & \beta_3 c_2 & \gamma_3 \end{bmatrix} \end{aligned} \tag{2}$$

The problem is to solve for the unknowns given the boundary invariants fixed. (See Goldman [JDG1990].)

Given a closed curve $\gamma$ on $\Sigma$, we obtain a function

$$tr(\gamma) : Def(\Sigma) \to \mathbb{R}_+$$

given by sending an equivalence class of projective structure $\mu$ to the trace of the conjugacy classes of holonomy of $\gamma$ corresponding to $\mu$. We can also define this for $P$ replacing $\Sigma$.

**Theorem**

*The trace functions have only positive values.*

By Positivstellensatz, we can show that trace functions are positive sums of monomials in defining functions of $D(\Sigma)$ multiplied by squares of rational functions.

**Proposition**

*Let $\alpha$ be a closed curve.*

- *The function $tr(\alpha) : Def(S_{p,q,r}) \to \mathbb{R}^+$ is a rational function of $s, t$.*
- *The function $tr(\alpha) : Def(P) \to \mathbb{R}^+$ is a rational function of $s, t, l_j, l_{j,2}$, $j = 1, 2, 3$.*
- *The function $tr(\alpha) : Def(\Sigma) \to \mathbb{R}^+$ is a rational function of $s_i, t_i, l_j, l_{j,2}$, $i = 1, 2, \ldots, 2g-2$, $j = 1, 2, \ldots, 3g-3$ and with denominators*

$$l_j - l_{j,2}, l_j + l_{j,2}, 1 - l_j l_{j,2}^2, 1 + l_j l_{j,2}^2, 1 - l_j^2 l_{j,2}, 1 + l_j^2 l_{j,2}, l_l + l_j l_{j,2}^2 l_k s$$

*for $c_l, c_k$ in a pair of pants containing $c_j$ and polynomials of gluing parameters $a_j, b_j$ (and one more longer denominator term with positive summands only).*

# Fock-Goncharov invariants

- Let $P_3^3$ denote the set of a triangle $A, B, C$ inscribed in a triangle with lines $a, b, c$ in $\mathbb{R}P^2$ mod out by $\mathrm{PGL}(3, \mathbb{R})$.

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$$X = \frac{f_a(v_B)f_b(v_C)f_c(v_A)}{f_a(v_C)f_b(v_A)f_c(v_B)}$$

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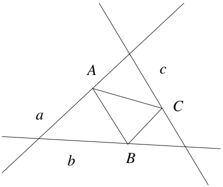


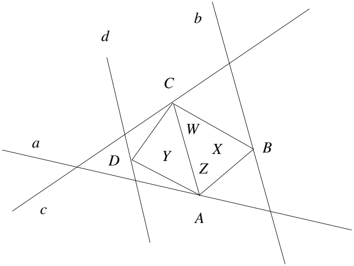


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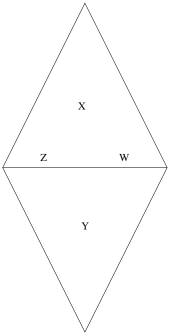
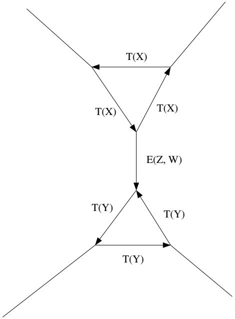
**Proof.**

Each marking gets an FG-invariant from a convex real projective structure. Conversely, FG-invariants on markings lets us construct a convex real projective structure. □

# Computing holonomy from Fock-Goncharov coordinates

Given a closed curve $\gamma$ on a framed convex real projective surface $S$, we can compute the holonomy by following the rule of multiplications. A caveat: these are not unimodular!



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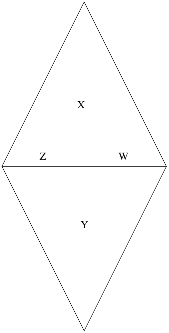
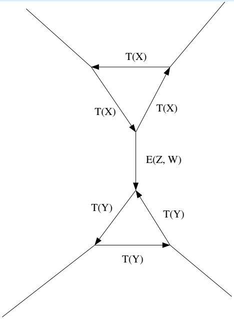
$$T(X) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ X & 1+X & 1 \end{bmatrix}, E(Z,W) = \begin{bmatrix} 0 & 0 & Z^{-1} \\ 0 & -1 & 0 \\ W & 0 & 0 \end{bmatrix}, \tag{3}$$



# Computing holonomy from Fock-Goncharov coordinates

**Theorem (Fock-Goncharov)**

*Let $Def(P)$ be the Fock-Goncharov space parameterized by cubit-root Fock-Goncharov coordinates. Let $h_\mu : \pi_1(S) \to \mathrm{SL}(3, \mathbb{R})$ the holonomy homomorphism associated with $\mu \in Def(P)$. The function $tr(h_\mu(\alpha)) : Def(P) \to \mathbb{R}^+$ is a rational function of cubit-root Fock-Goncharov coordinates which has only positive summands.*

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**Proof.**

Any curve can be freely homotoped to a product of $E(Z, W)T(X)$ or $E(Z, W)T(X)^{-1}$ for some $X, Z, W$s.

$$E(Z,W)T(X) = \begin{bmatrix} Z^{-1}X & Z^{-1}(1+X) & Z^{-1} \\ 0 & 1 & 1 \\ 0 & 0 & W^{-1} \end{bmatrix},$$
$$E(Z,W)T(X)^{-1} = \begin{bmatrix} Z^{-1}X & 0 & 0 \\ 1 & 1 & 0 \\ W & W(1+X^{-1}) & WX^{-1} \end{bmatrix} \tag{4}$$

□

# Traces in terms of Goldman coordinates

**Proposition (Positivity for a pair of pants)**

*$tr(\alpha) : Def(P) \to \mathbb{R}^+$ is a rational function of $l_i, l_{i,2}, s, t$ for $i = 1, 2, 3$ which has only positive summands.*

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**Proof.**

Fock-Goncharov invariants of $P$ can be computed from Goldman invariants $l_i, l_{i,2}, s, t$ as positive rational functions of $l_i, l_{i,2}, s, t$,

$$\frac{1}{l_2 l_{2,2}^2 l_3 + l_1 s}, \frac{1}{l_1 l_3 l_{3,2}^2 + l_2 s}, \frac{1}{l_2 + l_1 l_{1,2}^2 l_3 s}$$

with positive summands only. $\square$

### Positivity for orbifolds?

- Some experimental results show that for $\mathrm{CDef}(S_{p,q,r})$, the function $tr(\alpha)$ is a rational function of $l_i, \tau_i, s, t$ for $i = 1, 2, 3$ that has only positive summands.
- However, we do not have any proof. This observation began our study. (See the file "triangle5".)

# Semifields

- A *semifield* is $(\mathbb{S}, +, \cdot, 0, 1)$ where $+$ and $\cdot$ associative and commutative satisfying the distributivity law, and $0, 1 \in \mathbb{S}$ are identity elements, and
- $\mathbb{S}^* = \mathbb{S} - \{0\}$ is a group w.r.t $\cdot$. We write $a/b$ for $ab^{-1}$.
- 0 satisfies $0.x = x.0 = 0$ for all $x \in \mathbb{S}$.
- Furthermore, if $x + y = 0$, then $x = y = 0$. (the *zero-sum-free property*.)

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- Furthermore, if $x + y = 0$, then $x = y = 0$. (the *zero-sum-free property*.)

- Given an ordered abelian group $(\Lambda, +, <)$, add an extra element $-\infty$ so that $-\infty < x, \forall x \in \Lambda$.
- We define

$$\mathbb{T}_\Lambda = (\Lambda \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$$

where

$$a \oplus b = \max\{a, b\}, a \odot b = a + b \text{ if } a, b \in \Lambda \text{ or } -\infty \text{ if } a = -\infty \text{ or } b = -\infty.$$

The usual tropical field in the literature is $\mathbb{T}_\mathbb{R}$.

# Maslow dequantization

- Given $t \in (0,1)$, consider the map sending 0 to $-\infty$

$$\log_{\frac{1}{t}} : \mathbb{R}_+ \ni z \mapsto \log_{\frac{1}{t}}(z) = \left(\frac{-1}{\log t}\right) \log z \in \mathbb{R} \cup \{-\infty\}.$$

- The inverse function sending $-\infty$ to 0 is

$$\mathbf{D}_t := \log_{\frac{1}{t}}^{-1} : \mathbb{R} \cup \{-\infty\} \ni x \mapsto t^{-x} = \exp(x \log_{\frac{1}{t}}) \in \mathbb{R}_+.$$

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- The inverse function sending $-\infty$ to 0 is

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- The operations $+, \cdot$ are transformed by conjugation to

$$x \oplus_t y = \log_{\frac{1}{t}}(t^{-x} + t^{-y}), \quad x \odot_t y = \log_{\frac{1}{t}}(t^{-x} t^{-y}) = x + y.$$

- For every $t \in (0,1)$, $t$ induces a semifield structure on $\mathbb{R} \cup \{-\infty\}$ isomorphic to $\mathbb{R}_+$:

$$\mathbb{R}^t = (\mathbb{R} \cup \{-\infty\}, \oplus_t, \odot_t, -\infty, 0).$$

  The limit semifield is $\mathbb{T}_\mathbb{R}$.

## Compactification of semi-algebraic sets

- We define

$$\mathrm{Log}_{\frac{1}{t}} : \mathbb{R}_+^n \to (\mathbb{R} \cup \{-\infty\})^n, \qquad (x_1, \dots, x_n) \mapsto (\log_{\frac{1}{t}}(x_1), \cdots, \log_{\frac{1}{t}}(x_n))$$
$$(\mathrm{Log}_{\frac{1}{t}})^{-1} =: \mathbf{D}_t : (\mathbb{R} \cup \{-\infty\})^n \to \mathbb{R}_+^n, \quad (x_1, \dots, x_n) \mapsto (t^{-x_1}, \cdots, t^{-x_n}).$$

- Let $V \subset (\mathbb{R}_{>0})^n$ be a closed real semi-algebraic set. For $t \in (0,1)$, the *amoeba* of $V$ is
$$\mathcal{A}_t(V) = \{(\log_{\frac{1}{t}}(x_1), \dots, \log_{\frac{1}{t}}(x_n)) | (x_1, \dots, x_n) \in V\}.$$

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- We deform
$$W := \{(x, t) \in \mathbb{R}^n \times (0, \epsilon) | x \in \mathcal{A}_t(V)\}.$$
- Define
$$\mathcal{A}_0 = \pi(\bar{W} \cap (\mathbb{R}^n \times \{0\})) \subset \mathbb{R}^n.$$

**Theorem (Alessandrini)**

*Let $V \subset (\mathbb{R}_{>0})^n$ be a semi-algebraic set. Then the logarithmic limit set $\mathcal{A}_0(V) \subset \mathbb{R}^n$ is a polyhedral cone,* $\dim \mathcal{A}_0(V) \leq \dim V$*, and $\mathcal{A}_0(V) \cap \mathbb{Q}$ is dense in $\mathcal{A}_0(V)$.*

- We compactify $\mathbb{R}^n$ by adding the sphere at infinity

$$\mathbb{R}^n \ni x \mapsto \frac{x}{\sqrt{1 + ||x||^2}} \in D^n, \tag{5}$$

$$D^n \cong \mathbb{R}^n \cup \mathbb{S}^{n-1}. \tag{6}$$

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- We define

$$\partial V = (\mathcal{A}_0(V) - \{O\})/\sim \hookrightarrow \mathbb{S}^{n-1}.$$

- Given $0 < t_0 < 1$, the closure $\bar{V}$ of $\mathcal{A}_{t_0}(V)$ in $D^n$ equals

$$\bar{V} = \mathcal{A}_{t_0}(V) \cup \partial V.$$

- $\mathrm{CDef}(P)$ can be identified with a subset given by

$$D_P := \mathcal{D}_1 \times \mathcal{D}_2 \times \mathcal{D}_3 \times \mathbb{R}^2_+, \text{ where } \mathcal{D}_i := 0 < l_i \leq l_{i_2} \leq \frac{1}{l_i l_{i,2}}, \text{ for } i = 1, 2, 3.$$

- Let $\mathcal{F}$ be the set of generating trace family of $\pi_1(S)$. A *standard* family due to Lawton for a pair of pants $P$ is given by

$$f_a, f_{a^{-1}}, f_b, f_{b^{-1}}, f_{ab}, f_{a^{-1}b^{-1}}, f_{ab^{-1}}, f_{a^{-1}b}, f_{aba^{-1}b^{-1}}.$$

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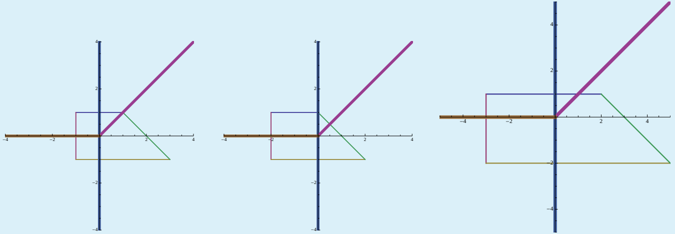

Figure: The Newton polytopes and their regions for $f_{ab^{-1}}$, $f_{a^{-1}b}$ and $f_{aba^{-1}b^{-1}}$. For each region, the extremal vertices for the Newton polytope of $f_{ab^{-1}}$, the one for $f_{a^{-1}b}$ and $f_{aba^{-1}b^{-1}}$ are linearly independent.

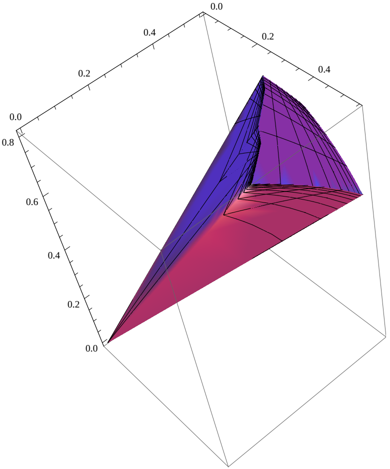


Figure: The tropical image of the map $(f_{ab^{-1}}, f_{a^{-1}b}, f_{aba^{-1}b^{-1}})$ which is a union of three cones with vertex the origin and the boundary arcs in the unit sphere.

**Theorem**

*Let* $\mathrm{CDef}(P)_{(l_1, l_{1,2}, l_2, l_{2,2}, l_3, l_{3,2})}$ *be the subspace of* $\mathrm{CDef}(P)$ *where the boundary invariants are fixed. Then its closure in the compactification of* $\mathrm{CDef}(P)$ *homeomorphic to a disk.*

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**Theorem**

*Given a small 2-orbifold $S(p,q,r)$ with $1/p+1/q+1/r<1$, suppose that all trace functions of $\mathrm{CDef}(S(p,q,r))$ is a positive rational function in Goldman coordinates $s,t$. Then the compactification of $\mathrm{CDef}(S(p,q,r))$ is homeomorphic to the closed unit ball $B^2$.*

This agrees with Cooper-Delp compactification as $\mathbb{R}P^2$ when we consider only the length functions.

# Some other results following the Bonahon-Dreyer approach

- Let $P$ be a pair of pants with a triangulation into two triangles. We take one where a boundary edge $l$ of a triangle ends in only one end.
- Letting the center invariants to be constant, we can vary six variables of the FG-coordinates.

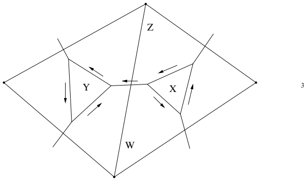


Figure: The diagram to compute the holonomy

# Some other results following the Bonahon-Dreyer approach

- Let $P$ be a pair of pants with a triangulation into two triangles. We take one where a boundary edge $l$ of a triangle ends in only one end.
- Letting the center invariants to be constant, we can vary six variables of the FG-coordinates.
- We fix all other invariants other than two $\omega$ and $\zeta$ on $l$.
- The tropical spectrum is

$$[0, -3(\omega+\zeta), -6(\omega+\zeta), \ldots] \sim [0, 1, 2, \ldots]$$

- Hence, there is a collapsing of the FG-coordinates.

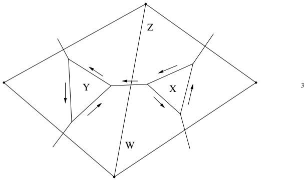


Figure: The diagram to compute the holonomy

# Understandable regions

- First, we look at FG-matrices that are of form $ET$ or $ET^{-1}$. The tropicalization of the matrix is: For $i = 1, 2, 3$, we have

$$M_{\zeta_i,\omega_i} := \begin{pmatrix} -2\zeta_i - \omega_i + 2x_i & -2\zeta_i - \omega_i + \max\{2x_i, -x_i\} & -2\zeta_i - \omega_i - x_i \\ -\infty & \zeta_i - \omega_i - x_i & \zeta_i - \omega_i - x_i \\ -\infty & -\infty & \zeta_i + 2\omega_i - x_i \end{pmatrix}, \tag{7}$$

$$N_{\zeta_i,\omega_i} := \begin{pmatrix} -2\zeta_i - \omega_i + x_i & -\infty & -\infty \\ \zeta_i - \omega_i + x_i & \zeta_i - \omega_i + x_i & -\infty \\ \zeta_i + 2\omega_i + x_i & \zeta_i + 2\omega_i + \max\{x_i, -2x_i\} & \zeta_i + 2\omega_i - 2x_i \end{pmatrix}. \tag{8}$$

- We restrict to considering the cone $C$ defined by the $(1, 1)$-entry of each matrix being larger than or equal to other terms in the matrix:

$$x_i > 0, \zeta_i < 0, \omega_i < 0 \tag{9}$$

is the cone we consider. For every $i$, we assume that this is true.

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- The another region is $C_-$

$$x_i < 0, \omega_i > 0, \zeta_i > 0 \tag{10}$$

This region is symmetric to the first one (9).

- Let $D$ be the deformation space. $D \subset \mathbb{R}^{-8\chi(S)}$. Let $\mathcal{C}$ denote the set of closed curves. We define function $F : D \to \mathbb{R}^{\mathcal{C}}$ given by $(F(\mu))_\alpha = f_\alpha(\mu)$.

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- Let $\mathcal{C}^{even}$ be the set of unoriented simple closed curves and let $\mathcal{C}^{odd} = \mathcal{C}$; the unit vectors in the 1-homology group. (representable by connected circles.)
- We define $F^{even} : D \to \mathbb{R}^{\mathcal{C}^{even}}$ and $F^{odd} : D \to \mathbb{R}^{\mathcal{C}^{odd}}$.

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- We define $F^{even} : D \to \mathbb{R}^{\mathcal{C}^{even}}$ and $F^{odd} : D \to \mathbb{R}^{\mathcal{C}^{odd}}$.
- The main task is to show the injectivity of $F : D \to \mathbb{R}^{\mathcal{C}}$.

**Proposition**

*The kernel of $F^{even} \times F^{odd} : D \to \mathbb{R}^{\mathcal{C}^{even}} \times \mathbb{R}^{\mathcal{C}}$ is same as the kernel of $F$.*

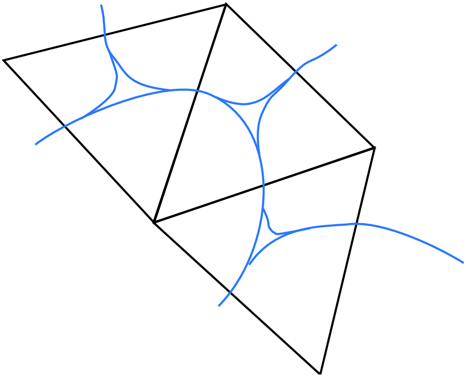

Figure: The train tracks and weight, and FG-invariants

**Conjecture**

- *$\mathcal{F} : D \to \mathbb{R}^{\mathcal{C}}$ sends a cone $C^{2E+F}$ in $D^{2E+F}$ to a cone of dimension $E + F + \dim H^1(\bar{S}, \partial\bar{S})$.*
- *It is never injective. The cone collapse by $E - \dim H^1(\bar{S}, \partial\bar{S})$.*
- *This gives us a set in the boundary of the compactification of dimension $E + F + \dim H^1(\bar{S}, \partial\bar{S}) - 1$.*
- *Morover, the antipodal cone $C_-$ maps to the cone of the same dimension.*

- This follows since $F^{odd}$ is homological. (following Bonahon-Dreyer.)