The ends of convex real projective manifolds and orbifolds

Suhyoung Choi

KAIST, Daejeon, MSRI, and UC Davis
email: schoi@math.kaist.ac.kr

Jointly with Yves Carrière and David Fried

UC Davis topology seminar April 28, 2015

Abstract

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- Among open real projective orbifolds that are topologically tame, we consider ones with radial ends and totally geodesic ends. We will present our work to classify these ends with some natural conditions.

Orbifolds

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By an *n*-dimensional orbifold, we mean a Hausdorff 2nd countable topological space with

- a fine open cover $\{U_i, i \in I\}$
- with models $(\tilde{U}_i, G_i)$ where G_i is a finite group acting on the $\tilde{U}_i \subset \mathbb{R}^n$, and
- a map $p_i : \tilde{U}_i \to U_i$ inducing $\tilde{U}_i/G_i \cong U_i$ where

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  - (compatibility) for each $i, j, x \in U_i \cap U_j$, there exists U_k with $x \in U_k \subset U_i \cap U_j$ and the inclusion $U_k \to U_i$ induces $\tilde{U}_k \to \tilde{U}_i$ with respect to $G_k \to G_i$.

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Good orbifolds

Our orbifolds are of form M/Γ for a simply connected manifold M and a discrete group Γ acting on M properly discontinuously.

Topology of our orbifolds

- Let $\mathcal{O}$ denote an n-dimensional orbifold with finitely many ends with end neighborhoods, closed $(n-1)$-dimensional orbifold times an open interval. (strongly tame).
- Equivalently, $\mathcal{O}$ has a compact suborbifold K so that $\mathcal{O} - K$ is a disjoint union $\Omega_i \times [0,1)$ for closed $n-1$-orbifolds Ω_i.

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Real projective and affine geometry

Real projective geometry

- Recall that the real projective space

$$\mathbb{RP}^n := P(\mathbb{R}^{n+1}) := \mathbb{R}^{n+1} - \{O\}/\sim \text{ under}$$

$$\vec{v} \sim \vec{w} \text{ iff } \vec{v} = s\vec{w} \text{ for } s \in \mathbb{R} - \{O\}.$$

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- $\mathrm{GL}(n+1, \mathbb{R})$ acts on $\mathbb{R}^{n+1}$ and $\mathrm{SL}_{\pm}(n+1, \mathbb{R})$ acts faithfully on $\mathbb{S}^n$.

Properly convex domain

- An affine subspace $\mathbb{R}^n$ can be identified with $\mathbb{RP}^n - V$ where V a hyperspace. Geodesics agree.
- $Aff(\mathbb{R}^n) = Aut(\mathbb{RP}^n - V)$.
- A *convex subset* of $\mathbb{RP}^n$ is a convex subset of an affine subspace.
- A *properly convex subset* of $\mathbb{RP}^n$ is a precompact convex subset of an affine subspace.
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Sphere version

- An open hemisphere is the affine subspace in $\mathbb{S}^n$ with boundary a hypersphere V. Now, the geometry is exactly the same as the above.
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Real projective structures on orbifolds

A discrete group Γ acts on a simply connected manifold M properly discontinuously.

A $\mathbb{R}P^n$-structure on M/Γ is given by

- an immersion $D : M \to \mathbb{R}P^n$ (*developing map*)
- equivariant with respect to a homomorphism $h : \Gamma \to \mathrm{PGL}(n+1, \mathbb{R})$. (*holonomy homomorphism*)

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- M an interior of a conic in $\mathbb{R}P^n$ of sign $(-, + \cdots, +)$. Discrete $\Gamma \subset PO(n, 1)$ and M/Γ is a hyperbolic orbifold and a convex $\mathbb{R}P^n$-orbifold.

- An $\mathbb{R}P^n$-structure on M/Γ is *convex* if D is a diffeomorphism to a convex domain

$$D(M) \subset A^n \subset \mathbb{R}P^n$$

- Identify $M = D(M)$ and Γ with its image under h.
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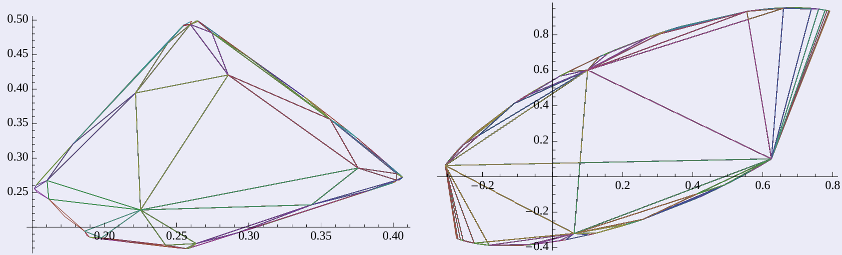

Figure: The developing images of convex $\mathbb{R}P^n$-structures on 2-orbifolds deformed from hyperbolic ones: $S^2(3, 3, 5)$ and $D^2(2, 7)$

Some useful facts on properly convex real projective orbifolds

- A properly convex domain has a Hilbert metric. Thus, a properly convex real projective orbifold admits a Hilbert metric. (Hilbert, Kobayashi)
- The interior of a unit sphere in an affine space has the hyperbolic metric as the Hilbert metric. (Beltrami-Klein) Thus, all hyperbolic manifolds admit convex real projective structures.

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- Some hyperbolic manifolds deform. (Kac-Viberg, Vinberg, Johnson-Millson, Cooper-Long-Thistlethwait)
- If the orbifold is closed, $\partial\Omega$ is C^1. If C^2, Ω is the interior of a unit sphere. and Ω/Γ is hyperbolic. (Benzécri)
- Convex real projective closed surfaces of genus > 1 is classified by Goldman by pairs-of-pants decomposition.
- For a closed manifold Ω/Γ, $\partial\Omega$ is strictly convex if and only if Γ is Gromov hyperbolic. (Benoist)

Dual real projective orbifolds

Dual domains

- An open convex cone C in $\mathbb{R}^{n+1}$ is *dual* to C^* in $\mathbb{R}^{n+1,*}$ if C^* is the set of of linear forms taking positive values on $\mathrm{Cl}(C) - \{O\}$.
- A convex open domain Ω in $P(\mathbb{R}^{n+1})$ is *dual* to Ω^* in $P(\mathbb{R}^{n+1*})$ if Ω corresponds to an open convex cone C and Ω^* to its dual C^*.

- Given a properly convex real projective n-orbifold Ω/Γ, there exists a dual one Ω^*/Γ^* with dual group given by

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- There exists a diffeomorphism $\Omega/\Gamma \leftrightarrow \Omega^*/\Gamma^*$. (Vinberg)
- Let $\mathcal{O} = \Omega/\Gamma$. Then $(\mathcal{O}^*)^* = \mathcal{O}$.
- The length spectrum determines closed $\mathcal{O}$ up to duality (Inkang Kim, Cooper-Delp)

Real projective structures on the ends

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Radial end (R-ends):

Each end E has an end neighborhood foliated by lines developing into lines ending at a common point. The space of leaves gives us the *end orbifold* Σ_E with a transverse real projective structure.

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Each end E has an end neighborhood foliated by lines developing into lines ending at a common point. The space of leaves gives us the *end orbifold* Σ_E with a transverse real projective structure.

Totally geodesic end (T-ends):

Each end has an end neighborhood completed by a closed totally geodesic orbifold of codim 1. The orbifold S_E is called an *ideal boundary* of the end or the end neighborhood. Clearly, it has a real projective structure.

Some definitions for radial ends

- A subdomain K of an affine subspace A^n in $\mathbb{R}P^n$ is said to be *horospherical* if it is strictly convex and the boundary ∂K is diffeomorphic to $\mathbb{R}^{n-1}$ and $\mathrm{bd} K - \partial K$ is a single point.
- K is *lens-shaped* if it is a convex domain in A^n and ∂K is a disjoint union of two strictly convex $(n-1)$-cells $\partial_+ K$ and $\partial_- K$.
- A *cone* is a domain D in A^n that has a point $v \in \mathrm{bd} D$ called a *cone-point* so that

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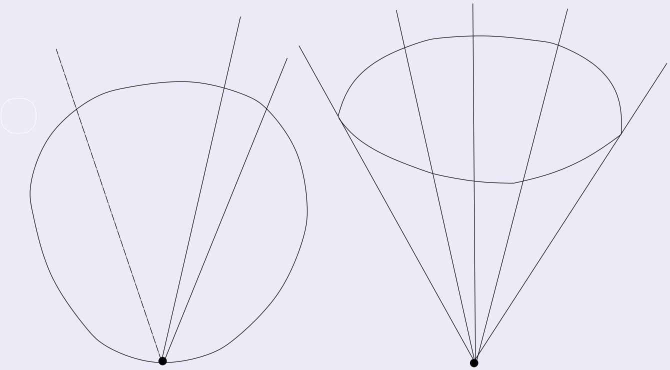
$$D = v * K - \{v\} \text{ for some } K \subset \mathrm{bd}D.$$
- A *cone D over* a lens-shaped domain L is a convex submanifold that contains L so that
$$D = v * \partial_+ L - \{v\}$$
for $v \in \mathrm{bd}D$ for a boundary component $\partial_+ L$ of ∂L and $\partial_+ L \subset \mathrm{bd}D$. (Every segment must meet $\partial_- L$.)

- We can allow one component $\partial_+ L$ be not smooth. In this case, we call these *generalized lens* and generalized lens-cone.

The universal covers of horospherical and lens shaped ends. The radial lines form cone-structures.



Definition on ends continued

- A *totally-geodesic subdomain* is a convex domain in a hyperspace. A *cone-over* a totally-geodesic domain A is a cone over a point x not in the hyperspace.

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$$C_1 + \cdots + C_m := \{v | v = c_1 + \cdots + c_m, c_i \in C_i\}.$$

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- A *join* of convex sets Ω_i in $\mathbb{R}P^n$ is given as

$$\Omega_1 * \cdots * \Omega_m := \Pi(C_1 + \cdots C_m)$$

where each C_i corresponds to Ω_i and these subspaces are independent. (We can relax this last condition)

Examples of deformations for orbifolds with radials ends

- Vinberg gave many examples for Coxeter orbifolds. (There are now numerous examples due to Benoist, Choi, Lee, Marquis.)
- Cooper-Long-Thistlethwaite also consider some hyperbolic 3-manifolds with ends.
- There is a census of small hyperbolic orbifolds with graph-singularity. (See the paper by D. Heard, C. Hodgson, B. Martelli, and C. Petronio [33].) S. Tillman constructed an example on $\mathbb{S}^3$ with a handcuff graph singularity.

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- These have lens type or horospherical ends by our theory to be presented.

End orbifold

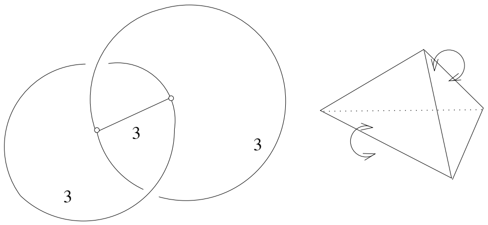

Figure: The handcuff graph

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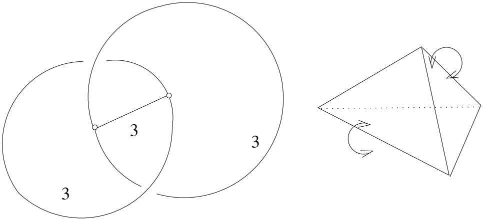

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Cusp ends

Let *M* be a complete hyperbolic manifolds with cusps. *M* is a quotient space of the interior Ω of a conic in $\mathbb{RP}^n$ or $\mathbb{S}^n$. Then the horoballs form the horospherical ends. Any end with a projective diffeomorphic end neighborhood is also called a *cusp*.

Back to Theory: p-ends, p-end neighborhood, p-end fundamental group

End fundamental group

- Given an end E of $\mathcal{O}$, a system of connected end neighborhoods $U_1 \supset U_2 \supset \cdots$ of $\mathcal{O}$ gives such a system $U_1' \supset U_2' \supset \cdots$ in $\tilde{\mathcal{O}}$.
- On each the end group $\Gamma_{\tilde{E}}$ acts. That is $U_i'/\Gamma_{\tilde{E}} \to U_i$, a homeomorphism.
- There are called these *proper pseudo-end neighborhood* in $\tilde{\mathcal{O}}$ and defines a *pseudo-end* $\tilde{E}$. (p-end nhbd, p-end from now on)

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Correspondences

$$\{E|E \text{ is an end of } \mathcal{O}\} \leftrightarrow \{\tilde{E}|\tilde{E} \text{ is a p-end of } \tilde{\mathcal{O}}\}/\pi_1(\mathcal{O})$$
$$\leftrightarrow \{\Gamma_{\tilde{E}}|\tilde{E} \text{ is a p-end of } \tilde{\mathcal{O}}\}/\pi_1(\mathcal{O}). \quad (1)$$

R-ends

A point $\mathbf{v}_{\tilde E} \in \mathbb{R}P^n$, the set of directions of lines from $\mathbf{v}_{\tilde E}$ form a sphere $\mathbb{S}^{n-1}_v$.

- Given a radial p-end $\tilde E$: $\mathbf{v}_{\tilde E}$,
$$R_{\mathbf{v}_{\tilde E}} =: \tilde\Sigma_{\tilde E} \subset \mathbb{S}^{n-1}_{\mathbf{v}_{\tilde E}}$$
the space of directions from $\mathbf{v}_{\tilde E}$ ending in $\tilde{\mathcal{O}}$.
- For radial end, $\Sigma_{\tilde E} := \tilde\Sigma_{\tilde E}/\Gamma_{\tilde E}$ is the end orbifold with the transverse real projective structure associated with E. (or write simply Σ_E)

T-ends

For a T-end, $\tilde S_{\tilde E}$ completes a closed p-end neighborhood. of a p-T-end. $S_{\tilde E} := \tilde S_{\tilde E}/\Gamma_{\tilde E}$ is a ideal boundary component corresponding to E. (or S_E.)

Real projective $n-1$-orbifolds associated with ends

Usual assumption

Let $\mathcal{O}$ a properly convex and strongly tame real projective orbifolds with radial or totally geodesic ends. The holonomy homomorphism is strongly irreducible. (The end fundamental group is of infinite index.)

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R-ends:

An R-end E has an end orbifold Σ_E admitting a real projective structure of dim $= n-1$. The structure is convex real projective one. The structure can be

- **PC**: properly convex,
- **CA**: complete affine, or
- **NPCC**: convex, not properly convex, not complete affine.

T-end

A totally geodesic end E has the ideal boundary orbifold S_E admitting a real projective structure of dim $= n - 1$. Here the structure is properly convex.

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Nonradial types (we do not study these)

- Convex ends – Geometrically finite, or infinite. (Even topologically wild?)
- ends that can be completed by lower dimensional strata– sometimes correspond to "geometrical Dehn surgeries". (hyperbolic Dehn surgery and other types also)
- Recent example: $P(S^+_{3\times 3})/\mathrm{SL}(3, \mathbb{Z})$ by Cooper for the space $S^+_{3\times 3}$ of positive definite 3×3-matrices. (Borel already studied these.)

Definitions for ends

Admissible group

An *admissible group* $\Gamma_{\tilde{E}}$ is a p-end fundamental group acting on

$$\tilde{\Sigma}_{\tilde{E}} = (\Omega_1 * \cdots * \Omega_k)^o$$

for strictly convex domains Ω_i of dimension $j_i \geq 0$ and is a finite extension of a finite product of $\mathbb{Z}^k \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_k$ for infinite hyperbolic groups $\mathbf{\Gamma}_i$ where

- each Γ_i acts cocompactly on an open strictly convex domain Ω_i and trivially on Ω_j for $j \neq i$ and
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Question

Can we improve the strict convexity to the proper convexity?

Classification of the PC-ends

Definition: umec

Let $\tilde{E}$ be a properly convex R-end with $\tilde{\Sigma}_{\tilde{E}} = \Omega$. The end fundamental group $\mathbf{\Gamma}_{\tilde{E}}$ satisfies the *uniform middle eigenvalue condition* (umec) if if every $g \in \mathbf{\Gamma}_{\tilde{E}}$ satisfies

$$K^{-1}\mathrm{length}_{\Omega}(g) \leq \log\left(\frac{\lambda_1(g)}{\lambda_{\mathbf{v}_{\tilde{E}}}(g)}\right) \leq K\mathrm{length}_{\Omega}(g), \qquad (2)$$

for the largest eigenvalue modulus $\lambda_1(g)$ of g and the eigenvalue of g at $\mathbf{v}_{\tilde{E}}$ for g in $\mathbf{\Gamma}_E$. Also, the same has to hold for each factor $\mathrm{Cl}(\Omega_i)$ where the maximal norm of the eigenvalue in the factor is used.

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There is a dual definition for T-ends.

Main results

Theorem 1 (Main result for PC R-ends)

Let $\mathcal{O}$ be a real projective orbifold with usual property.

- *Suppose that the end holonomy group of a properly convex R-end $\tilde{E}$ satisfies the uniform middle eigenvalue condition.*

Then $\tilde{E}$ is of generalized lens type. If we assume only weak uniform middle eigenvalue condition, then the end can also be quasi-lens type.

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Theorem 3.1

Let $\mathcal{O}$ satisfy the usual condition. Let $\Gamma_{\tilde{E}}$ the holonomy group of a properly convex p-R-end $\tilde{E}$. Assume that $\mathcal{O}$ satisfies the triangle condtion or $\tilde{E}$ is virtually factorizable. Then the following statements are equivalent:

(i) *$\Gamma_{\tilde{E}}$ is of lens-type.*

(ii) *$\Gamma_{\tilde{E}}$ satisfies the uniform middle eigenvalue condition.*

Theorem 2

Let $\mathcal{O}$ be as usual. Let $\tilde{S}_{\tilde{E}}$ be a totally geodesic ideal boundary of a totally geodesic p-T-end $\tilde{E}$ of $\tilde{\mathcal{O}}$. Then the following conditions are equivalent:

- (i) *$\tilde{E}$ satisfies the uniform middle-eigenvalue condition.*
- (ii) *$\tilde{S}_{\tilde{E}}$ has a lens-neighborhood in an ambient open manifold containing $\tilde{\mathcal{O}}$ and hence $\tilde{E}$ has a lens-type p-end neighborhood in $\tilde{\mathcal{O}}$.*

Nonproperly convex (NPCC) ends

- Let $\tilde{E}$ be a p-end of $\mathcal{O}$ with $\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathbf{v}_{\tilde{E}}}$ is convex but not properly convex and not complete affine, and let U the corresponding end neighborhood in $\mathbb{S}^n$ with the end vertex $\mathbf{v}_{\tilde{E}}$.

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- $\tilde{\Sigma}_{\tilde{E}}$ is foliated by affine spaces (or open hemispheres) of dimension i. with common boundary $\mathbb{S}^{i-1}_\infty$. The space of i-dimensional hemispheres with boundary $\mathbb{S}^{i-1}_\infty$ equals projective $\mathbb{S}^{n-i-1}$. The space of i-dimensional leaves form a properly convex domain K in $\mathbb{S}^{n-i-1}$.

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- Now going $\mathbb{S}^n$. Each hemiphere $H^i \subset \mathbb{S}^{n-1}_{\mathbf{v}_{\tilde{E}}}$ with $\partial H^i = \mathbb{S}^{i-1}_\infty$ corresponds to H^{i+1} in $\mathbb{S}^n$ whose common boundary $\mathbb{S}^i_\infty$ that contains $\mathbf{v}_{\tilde{E}}$. Note $\mathbb{S}^i_\infty$ is $h(\pi_1(E))$-invariant.

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- We let N be the subgroup of $h(\pi_1(E))$ of elements inducing trivial actions on $\mathbb{S}^{n-i-1}$.

Proposition 4.1

Let $\tilde{E}$ be a NPCC p-end of a properly convex n-orbifold $\mathcal{O}$ with usual conditions. Let $\hat{h}_{\tilde{E}} : \pi_1(\tilde{E}) \to \mathbf{Aut}(\mathbb{S}^{n-1})$ be the associated holonomy homomorphism for the corresponding end vertex $\mathbf{v}_{\tilde{E}}$. Then

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- $\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathbf{v}_{\tilde{E}}}$ *is foliated by complete affine subspaces of dimension i, $i > 0$.*
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- $\tilde{\Sigma}_{\tilde{E}} \subset \mathbb{S}^{n-1}_{\mathbf{v}_{\tilde{E}}}$ *is foliated by complete affine subspaces of dimension i, $i > 0$.*
- *The space K of leaves is a properly convex domain of dimension $n - 1 - i$.*
- $h(\pi_1(E))$ *acts on the great sphere $\mathbb{S}^{i-1}_{\infty}$ of dimension $i - 1$ in $\mathbb{S}^{n-1}_{\mathbf{v}_{\tilde{E}}}$.*
- *There exists an exact sequence*

$$1 \rightarrow N \rightarrow \pi_1(E) \rightarrow N_K \rightarrow 1$$

where N acts trivially on K and $N_K \subset Aut(K)$.

Some examples of NPCC p-ends: the join and the quasi-join

Lens part $v_1 * L$ where L is a properly open convex domain in a hyperspace S_1' outside v_1. Let Γ_1 acts on v_1 and L. L/Γ_1 is compact. (coming from some lens-type end) Assume $v_1 * L \subset S_1^{n-i_0-1}$ for a subspace.

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Horosphere Let H be horosphere with vertex v_2 in a subspace $S_2^{i_0+1}$ with Γ_2 act on it. $\partial H/\Gamma_2$ is a compact suborbifold.

Complementary We embed these in subspaces of $\mathbb{S}^n$ where

$$v_1 * L \subset S_1, H \subset S_2, \text{ so that } S_1 \cap S_2 = \{v_1 = v_2\}, S_1' \cap S_2 = \emptyset.$$

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Lens part $v_1 * L$ where L is a properly open convex domain in a hyperspace S'_1 outside v_1. Let Γ_1 acts on v_1 and L. L/Γ_1 is compact. (coming from some lens-type end) Assume $v_1 * L \subset S_1^{n-i_0-1}$ for a subspace.

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Join Obtain a join $(v_1 * L) * H$. Extend Γ_2 trivially on S_1. Extend Γ_1 to an action on S_2 normalizing Γ_2.

Example of quasi-join (not definition)

Joined action Find an infinite cyclic group action by g fixing every points S_1' and S_2. They correspond to different eigenspaces of g of eigenvalues $\lambda_1, \lambda_2, \lambda_1 < \lambda_2$. Then $\Gamma_1 \times \Gamma_2 \times \langle g \rangle$ acts on the join.

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Quasi-join We mutiply g by a unipotent translation T in S_2 towards v_2. Then $\Gamma_1 \times \Gamma_2 \times \langle g \rangle$ now acts on properly convex domain whose closure meets S_1 at v_2 only. For example, for $(\lambda < 1, k > 0)$,

$$g := \begin{pmatrix} \lambda^3 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & k & 0 & \frac{1}{\lambda} \end{pmatrix}, n := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & v & 1 & 0 \\ 0 & \frac{v^2}{2} & v & 1 \end{pmatrix}$$

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p-end The group will act on a properly convex p-end neighborhood.

Definition 3

We define $\lambda_1(g)$ to equal to the largest norm of the eigenvalue of g whose Jordan-form invariant subspace meets $\mathbb{S}^n - \mathbb{S}^{i_0}_\infty$, and $\lambda_{\mathbf{v}_{\tilde{E}}}(g)$ the eigenvalue of g at $\mathbf{v}_{\tilde{E}}$. The *weak middle-eigenvalue condition* means

$$\lambda_1(g) \geq \lambda_{\mathbf{v}_{\tilde{E}}}(g) \text{ for every } g \in \mathbf{\Gamma}_{\tilde{E}}.$$

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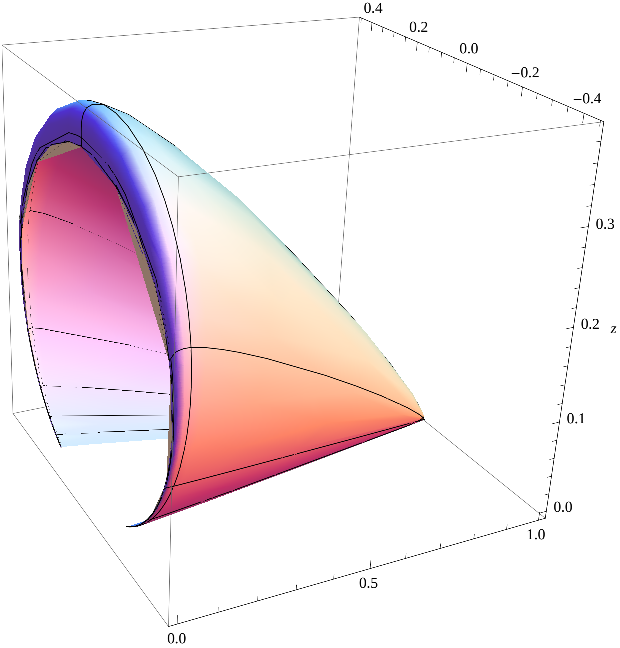
Theorem 4

Let $\Sigma_{\tilde{E}}$ be the end orbifold of a NPCC radial p-end $\tilde{E}$ of a strongly tame properly convex n-orbifold $\mathcal{O}$ satisfying usual conditions. Let $\mathbf{\Gamma}_{\tilde{E}}$ be the end fundamental group. We suppose that

- *$\mathbf{\Gamma}_{\tilde{E}}$ satisfies the weak middle-eigenvalue condition.*
- *Suppose that a virtual center of $\mathbf{\Gamma}_{\tilde{E}}$ maps to a Zariski dense group in* **Aut**(K) *for the space K of complete affine leaves of $\tilde{\Sigma}_{\tilde{E}}$.*

Then there exists a finite cover $\Sigma_{E'}$ of $\Sigma_{\tilde{E}}$ so that E' is a quasi-join of a properly convex totally geodesic R-ends and a cusp R-end.



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