Geometric structures on 2-orbifolds

Lie groups and geometry I

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Lectures at KAIST

Outline

- Geometries
  - Euclidean geometry
  - Spherical geometry
  - Affine geometry
  - Projective geometry
  - Conformal geometry: Poincare extensions
  - Hyperbolic geometry
    - Lorentz group
    - Geometry of hyperbolic space
    - Beltrami-Klein model
    - Conformal ball model
    - The upper-half space model
  - Discrete groups: examples
    - Discrete group actions
    - Convex polyhedrons
    - Side pairings and the fundamental theorem
    - Crystallographic groups

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Euclidean geometry

- The Euclidean space is $\mathbb{R}^n$ and the group $Isom(\mathbb{R}^n)$ of rigid motions is generated by $O(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.
- A system of linear equations gives us a subspace (affine or linear)
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Spherical geometry

- Let us consider the unit sphere $\mathbf{S}^n$ in the Euclidean space $\mathbb{R}^{n+1}$.
- Many great spheres exist and they are subspaces... (They are given by homogeneous system of linear equations in $\mathbb{R}^{n+1}$.)
- The lines are replaced by great circles and lengths and angles are also replaced.
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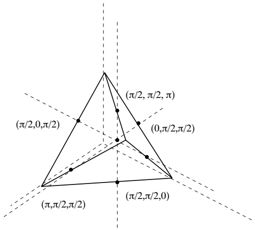
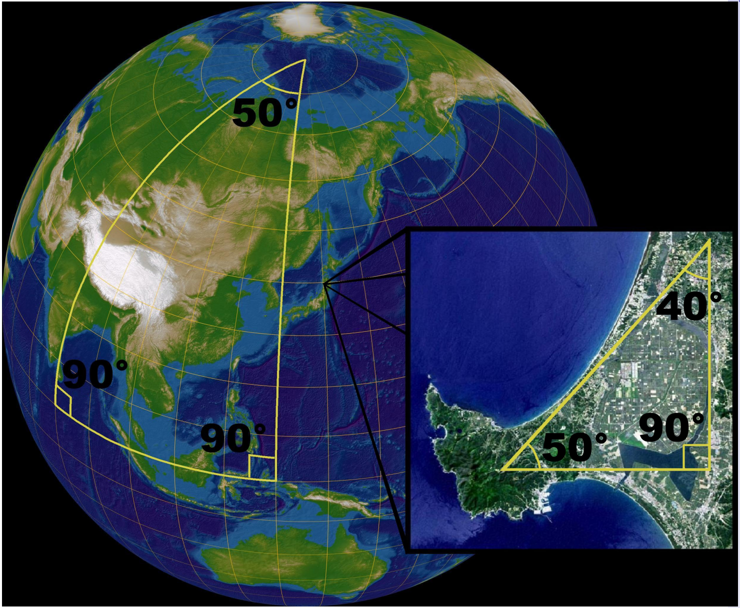
Spherical trigonometry

- Many spherical triangle theorems exist... http://mathworld.wolfram.com/SphericalTrigonometry.html
- Such a triangle is classified by their angles $\theta_0, \theta_1, \theta_2$ satisfying

$$\theta_0 + \theta_1 + \theta_2 > \pi \quad (1)$$
$$\theta_i < \theta_{i+1} + \theta_{i+2} - \pi, i \in \mathbb{Z}_3. \quad (2)$$



- 
50°
90°
90°
40°
50°
90°

Affine geometry

- A vector space $\mathbb{R}^n$ becomes an affine space by forgetting the origin.
- An affine transformation of $\mathbb{R}^n$ is one given by $x \mapsto Ax + b$ for $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. This notion is more general than that of rigid motions.
- The Euclidean space $\mathbb{R}^n$ with the group $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$ of affine transformations form the affine geometry.
- Of course, angles and lengths do not make sense. But the notion of lines exists.
- The set of three points in a line has an invariant based on ratios of lengths.

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Projective geometry

- Projective geometry was first considered from fine art.
- Desargues (and Kepler) first considered points at infinity.
- Poncelet first added infinite points to the euclidean plane.
- Projective transformations are compositions of perspectivities. Often, they send finite points to infinite points and vice versa. (i.e., two planes that are not parallel).
- The added points are same as ordinary points up to projective transformations.

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- Lines have well defined infinite points and are really circles topologically.
- Some notions lose meanings. However, many interesting theorems can be proved. Duality of theorems plays an interesting role.
- See for an interactive course: http://www.math.poly.edu/courses/projective_geometry/
- and http://demonstrations.wolfram.com/TheoremeDePappusFrench/, http://demonstrations.wolfram.com/TheoremeDePascalFrench/, http://www.math.umd.edu/~wphooper/pappus9/pappus.html, http://www.math.umd.edu/~wphooper/pappus/

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- Formal definition with topology is given by Felix Klein using homogeneous coordinates.
- The projective space $\mathbb{R}P^n$ is $\mathbb{R}^{n+1} - \{O\}/\sim$ where $\sim$ is given by $v \sim w$ if $v = sw$ for $s \in \mathbb{R}$.
- Each point is given a homogeneous coordinates: $[v] = [x_0, x_1, ..., x_n]$.
- The projective transformation group $\mathrm{PGL}(n+1, \mathbb{R}) = GL(n+1, \mathbb{R})/\sim$ acts on $\mathbb{R}P^n$ by each element sending each ray to a ray using the corresponding general linear maps.
- Here, each element of g of $\mathrm{PGL}(n+1, \mathbb{R})$ acts by $[v] \mapsto [g'(v)]$ for a representative g' in $GL(n+1, \mathbb{R})$ of g. Also any coordinate change can be viewed this way.

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- The affine geometry can be "imbedded": $\mathbb{R}^n$ can be identified with the set of points in $\mathbb{R}P^n$ where x_0 is not zero, i.e., the set of points $\{[1, x_1, x_2, ..., x_n]\}$. This is called an affine patch. The subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ fixing $\mathbb{R}^n$ is precisely $Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \cdot \mathbb{R}^n$.
- The subspace of points $\{[0, x_1, x_2, ..., x_n]\}$ is the complement homeomorphic to $\mathbb{R}P^{n-1}$. This is the set of infinities, i.e., directions in $\mathbb{R}P^n$.
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- A subspace is the set of points whose representative vectors satisfy a homogeneous system of linear equations. The subspace in $\mathbb{R}^{n+1}$ corresponding to a projective subspace in $\mathbb{R}P^n$ in a one-to-one manner while the dimension drops by 1.
- The independence of points are defined. The dimension of a subspace is the maximal number of independent set minus 1.
- A hyperspace is given by a single linear equation. The complement of a hyperspace can be identified with an affine space.
- A line is the set of points $[v]$ where $v = sv_1 + tv_2$ for $s, t \in \mathbb{R}$ for the independent pair v_1, v_2. Actually a line is $\mathbb{R}P^1$ or a line $\mathbb{R}^1$ with a unique infinity.

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- Cross ratios of four points on a line (x, y, z, t). There is a unique coordinate system so that $x = [1, 0], y = [0, 1], z = [1, 1], t = [b, 1]$. Thus $b = b(x, y, z, t)$ is the cross-ratio.
- If the four points are on $\mathbb{R}^1$, the cross ratio is given as

$$(x, y; z, t) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

if we can write

$$x = [1, z_1], y = [1, z_2], z = [1, z_3], t = [1, z_4]$$

- One can define cross ratios of four hyperplanes meeting in a projective subspace of codimension 2.
- For us $n = 2$ is important. Here we have a familiar projective plane as topological type of $\mathbb{R}P^2$, which is a Mobius band with a disk filled in at the boundary. http://www.geom.uiuc.edu/zoo/toptype/pplane/cap/

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Conformal geometry

- Reflections of $\mathbb{R}^n$. The hyperplane $P(a, t)$ given by $a \cdot x = t$. Then $\rho(x) = x + 2(t - a \cdot x)a$.
- Inversions. The hypersphere $S(a, r)$ given by $|x - a| = r$. Then $\sigma(x) = a + (\frac{r}{|x-a|})^2(x - a)$.
- We can compactify $\mathbb{R}^n$ to $\hat{\mathbb{R}}^n = \mathbf{S}^n$ by adding infinity. This can be accomplished by a stereographic projection from the unit sphere $\mathbf{S}^n$ in $\mathbb{R}^{n+1}$ from the northpole $(0, 0, ..., 1)$. Then these reflections and inversions induce conformal homeomorphisms.

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- The group of transformations generated by these homeomorphisms is called the Mobius transformation group.
- They form the conformal transformation group of $\hat{\mathbb{R}}^n = \mathbf{S}^n$.
- For $n = 2$, $\hat{\mathbb{R}}^2$ is the Riemann sphere $\hat{\mathbb{C}}$ and a Mobius transformation is a either a fractional linear transformation of form

$$z \mapsto \frac{az+b}{cz+d}, ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

or a fractional linear transformation pre-composed with the conjugation map $z \mapsto \bar{z}$.
- In higher-dimensions, a description as a sphere of null-lines and the special Lorentizian group exists.

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Poincare extensions

- We can identify E^{n-1} with $E^{n-1} \times \{O\}$ in E^n.
- We can extend each Mobius transformation of $\hat{E}^{n-1}$ to $\hat{E}^n$ that preserves the upper half space U: We extend reflections and inversions in the obvious way.
- The Mobius transformation of $\hat{E}^n$ that preserves the open upper half spaces are exactly the extensions of the Mobius transformations of $\hat{E}^{n-1}$.
- $M(U^n) = M(\hat{E}^{n-1})$.
- We can put the pair $(U^n, \hat{E}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobius transformation.
- Thus, $M(U^n)$ is isomorphic to $M(\mathbf{S}^{n-1})$ for the boundary sphere.

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- We can extend each Mobius transformation of $\hat{E}^{n-1}$ to $\hat{E}^n$ that preserves the upper half space U: We extend reflections and inversions in the obvious way.
- The Mobius transformation of $\hat{E}^n$ that preserves the open upper half spaces are exactly the extensions of the Mobius transformations of $\hat{E}^{n-1}$.
- $M(U^n) = M(\hat{E}^{n-1})$.
- We can put the pair $(U^n, \hat{E}^{n-1})$ to $(B^n, \mathbf{S}^{n-1})$ by a Mobius transformation.
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Lorentzian geometry

- A hyperbolic space H^n is defined as a complex Riemannian manifold of constant curvature equal to -1.
- Such a space cannot be realized as a submanifold in a Euclidean space of even very large dimensions.
- But it is realized as a "sphere" in a Lorentzian space.
- A Lorentzian space is $\mathbb{R}^{1,n}$ with an inner product

$$x \cdot y = -x_0 y_0 + x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n y_n.$$

- A Lorentzian norm $||x|| = (x \cdot y)^{1/2}$, a positive, zero, or positive imaginary number.
- One can define Lorentzian angles.
- The null vectors form a light cone divide into positive, negative cone, and 0.
- Space like vectors and time like vectors and null vectors.
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Lorentz group

- A Lorentzian transformation is a linear map preserving the inner-product.
- For J the diagonal matrix with entries $-1, 1, ..., 1$, $A^t J A = J$ iff A is a Lorentzian matrix.
- A Lorentzian transformation sends time-like vectors to time-like vectors. It is either positive or negative.
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Hyperbolic space

- Given two positive time-like vectors, there is a time-like angle
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- A hyperbolic space is an upper component of the submanifold defined by $||x||^2 = -1$ or $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$. This is a subset of a positive cone.
- Topologically, it is homeomorphic to $\mathbb{R}^n$. Minkowsky model
- One induces a metric from the Lorentzian space which is positive definite.
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- A hyperbolic line is an intersection of H^n with a time-like two-dimensional vector subspace.
- The hyperbolic sine law, The first law of cosines, The second law of cosines...
- One can assign any interior angles to a hyperbolic triangle as long as the sum is less than π.
- One can assign any lengths to a hyperbolic triangle.
- The triangle formula can be generalized to formula for quadrilateral, pentagon, hexagon.
- Basic philosophy here is that one can push the vertex outside and the angle becomes distances between lines. (See Ratcliffe, `http://online.redwoods.cc.ca.us/instruct/darnold/staffdev/Assignments/sinandcos.pdf`)

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- hyperbolic law of sines:

$$\sin A/\sinh a = \sin B/\sinh b = \sin C/\sinh c$$

- hyperbolic law of cosines:

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos C$$

$$\cos C = (\cosh a \cosh b - \cosh c)/\sinh a \sinh b$$

$$\cosh c = (\cos A \cos B + \cos C)/\sin A \sin B$$

Beltrami-Klein models of hyperbolic geometry

- Beltrami-Klein model is directly obtained from the hyperboloid model.
- $d_k(P, Q) = 1/2 \log |(AB, PQ)|$ where A, P, Q, B are on a segment with endpoints A, B and

$$(AB, PQ) = \left| \frac{AP}{BP} \frac{BQ}{AQ} \right|.$$

- There is an imbedding from H^n onto an open ball B in the affine patch $\mathbb{R}^n$ of $\mathbb{R}P^n$. This is standard radial projection $\mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$.
- B can be described as a ball of radius 1 with center at O.
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- The metric is induced on B. This is precisely the metric given by the log of the cross ratio. Note that $\lambda(t) = (\cosh t, \sinh t, 0, ..., 0)$ define a unit speed geodesic in H^n. Under the Riemannian metric, we have $d(e_1, (\cosh t, \sinh t, 0, ..., 0)) = t$ for t positive.
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- Beltrami-Klein model is nice because you can see outside. The outside is the anti-de Sitter space http://en.wikipedia.org/wiki/Anti_de_Sitter_space
- Also, we can treat points outside and inside together.
- Each line (hyperplane) in the model is dual to a point outside. (i.e., orthogonal by the Lorentzian inner-product) A point in the model is dual to a hyperplane outside. Infact any subspace of dimenstion i is dual to a subspace of dimension $n - i - 1$ by orthogonality.
- For $n = 2$, the duality of a line is given by taking tangent lines to the disk at the endpoints and taking the intersection.
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The conformal ball model (Poincare ball model)

- The stereo-graphic projection H^n to the plane P given by $x_0 = 0$ from the point $(-1, 0, ..., 0)$.
- The formula for the map $\kappa : H^n \to P$ is given by

$$\kappa(x) = \left(\frac{y_1}{1 + y_0}, ..., \frac{y_n}{1 + y_0}\right),$$

where the image lies in an open ball of radius 1 with center O in P. The inverse is given by

$$\zeta(x) = \left(\frac{1 + |x|^2}{1 - |x|^2}, \frac{2x_1}{1 - |x|^2}, ..., \frac{2x_n}{1 - |x|^2},\right).$$

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- We show

$$\cosh d_B(x, y) = 1 + \frac{2|x - y|^2}{(1 - |x|^2)(1 - |y|^2)},$$

  and inversions acting on B preserves the metric. Thus, the group of Mobius transformations of B preserve metric.
- The corresponding Riemannian metric is $g_{ij} = 2\delta_{ij}/(1 - |x|^2)^2$.
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The upper-half space model.

- Now we put B to U by a Mobius transformation. This gives a Riemannian metric constant curvature -1.
- We have by computations $\cosh d_U(x, y) = 1 + |x - y|^2/2x_n y_n$ and the Riemannian metric is given by $g_{ij} = \delta_{ij}/x_n^2$. Then $I(U) = M(U) = M(E^{n-1})$.
- Geodesics would be arcs on lines or circles perpendicular to E^{n-1}.
- Since $\hat{E}^1$ is a circle and $\hat{E}^2$ is the complex sphere, we obtain $Isom^+(B^2) = PSL(2, \mathbb{R})$ and $Isom^+(B^3) = PSL(2, \mathbb{C})$.

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- Orientation-preserving isometries of hyperbolic plane can have at most one fixed point. elliptic, hyperbolic, parabolic.

$$z \mapsto e^{i\theta}, z \mapsto az, a \neq 1, a \in \mathbb{R}^+, z \mapsto z + 1$$

- Isometries of a hyperbolic space: loxodromic, hyperbolic, elliptic, parabolic.
- Up to conjugations, they are represented as Mobius transformations which have forms
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Discrete groups and discrete group actions

- A discrete group is a group with a discrete topology. (Usually a finitely generated subgroup of a Lie group.) Any group can be made into a discrete group.
- We have many notions of a group action $\Gamma \times X \to X$:
  - The action is effective, is free
  - The action is *discrete* if Γ is discrete in the group of homeomorphisms of X with compact open topology.
  - The action has *discrete orbits* if every x has a neighborhood U so that the orbit points in U is finite.
  - The action is *wandering* if every x has a neighborhood U so that the set of elements γ of Γ so that $\gamma(U) \cap U \neq \emptyset$ is finite.
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- discrete action < discrete orbit < wandering < properly discontinuous. This is a strict relation (Assuming X is a manifold.)
- The action is wandering and free and gives manifold quotient (possibly non-Hausdorff)
- The action of Γ is free and properly discontinuous if and only if X/Γ is a manifold quotient (Hausdorff) and $X \to X/\Gamma$ is a covering map.
- Γ a discrete subgroup of a Lie group G acting on X with compact stabilizer. Then Γ acts properly discontinuously on X.
- A complete (X, G) manifold is one isomorphic to X/Γ.
- Suppose X is simply-connected. Given a manifold M the set of complete (X, G)-structures on M up to (X, G)-isotopies are in one-to-one correspondence with the discrete representations of $\pi(M) \to G$ up to conjugations.

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Examples

- $\mathbb{R}^2 - \{O\}$ with the group generated by $g_1 : (x, y) \to (2x, y/2)$. This is a free wondering action but not properly discontinuous.
- $\mathbb{R}^2 - \{O\}$ with the group generated by $g : (x, y) \to (2x, 2y)$. (free, properly discontinuous.)
- The modular group $PSL(2, \mathbb{Z})$ the group of Mobius transformations or isometries of hyperbolic plane given by $z \mapsto \frac{az+b}{cz+d}$ for integer a, b, c, d and $ad - bc = 1$. `http://en.wikipedia.org/wiki/Modular_group`. This is not a free action.

Convex polyhedrons

Suppose that X is a space where a Lie group G acts effectively and transitively. Furthermore, suppose X has notions of m-planes. An *m-plane* is an element of a collection of submanifolds of X of dimension m so that given generic $m+1$ point, there exists a unique one containing them. We require also that every 1-plane contains geodesic between any two points in it. Obviously, we assume that elements of G sends m-planes to m-planes. (For complex hyperbolic spaces, such notion seemed to be absent.)

We also need to assume that X satisfies the increasing property that given an m-plane and if the generic $m+1$-points in it, lies in an n-plane for $n \geq m$, then the entire m-plane lies in the n-plane.

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For example, any geometry with models in $\mathbb{R}P^n$ and G a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ has a notion of m-planes. Thus, hyperbolic, euclidean, spherical, and projective geometries has notions of m-planes. Conformal geometry may not have such notions since generic pair of points have infinitely many circles through them.

A *convex subset* of X is a subset such that for any pair of points, there is a unique geodesic segment between them and it is in the subset. For example, a pair of antipodal point in $\mathbf{S}^n$ is convex.

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Assume that X is either $\mathbf{S}^n$, $\mathbb{R}^n$, H^n, or $\mathbb{R}P^n$ with Lie groups acting on X. Let us state some facts about convex sets:

- The dimension of a convex set is the least integer m such that C is contained in a unique m-plane $\hat{C}$ in X.
- The interior C^o, the boundary ∂C are defined in $\hat{C}$.
- The closure of C is in $\hat{C}$. The interior and closures are convex. They are homeomorphic to an open ball and a contractible domain of dimension equal to that of $\hat{C}$ respectively.
- A side C is a nonempty maximal convex subset of ∂C.
- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in X.

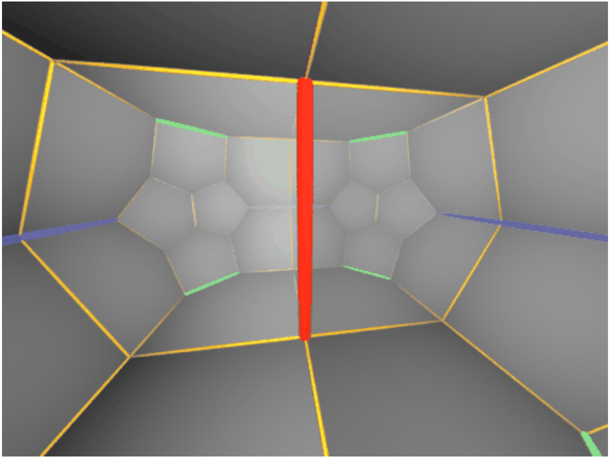
Convex polytopes

- A side C is a nonempty maximal convex subset of ∂C.
- A convex polyhedron is a nonempty closed convex subset such that the set of sides is locally finite in H^n.
- A polytope is a convex polyhedron with finitely many vertices and is the convex hull of its vertices inH^n.
- A polyhedron P in H^n is a generalized polytope if its closure is a polytope in the affine patch. A generalized polytope may have ideal vertices.

Examples of Convex polytopes

- A compact simplex: convex hull of $n+1$ points in H^n.
- Start from the origin expand the infinitesimal euclidean polytope from an interior point radially. That is a map sending $x \to sx$ for $s > 0$ and x is the coordinate vector of an affine patch using in fact any vector coordinates. Thus for any Euclidean polytope, we obtain a one parameter family of hyperbolic polytopes.
-

Regular dodecahedron with all edge angles $\pi/2$



Fundamental domain of discrete group action

- Let Γ be a group acting on X.
- A *fundamental domain* for Γ is an open domain F so that $\{gF|g \in \Gamma\}$ is a collection of disjoint sets and their closures cover X.
- The fundamental domain is locally finite if the above closures are locally finite.
- The *Dirichlet domain* for $u \in X$ is the intersection of all $H_g(u) = \{x \in X|d(x,u) < d(x,gu)\}$. Under nice conditions, $D(u)$ is a convex fundamental polyhedron.
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Tessellations

- A tessellation of X is a locally-finite collection of polyhedra covering X with mutually disjoint interiors.
- Convex fundamental polyhedron provides examples of exact tessellations.
- If P is an exact convex fundamental polyhedron of a discrete group Γ of isometries acting on X, then Γ is generated by $\Phi = \{g \in \Gamma | P \cap g(P) \text{ is a side of } P\}$.

Side pairings and Poincare fundamental polyhedron theorem

- Given a side S of an exact convex fundamental domain P, there is a unique element g_S such that $S = P \cap g_S(P)$. And $S' = g_S^{-1}(S)$ is also a side of P.
- $g_{S'} = g_S^{-1}$ since $S' = P \cap g_S^{-1}$.
- Γ-side-pairing is the set of g_S for sides S of P.
- The equivalence class at P is generated by $x \cong x'$ if there is a side-pairing sending x to x' for $x, x' \in P$.
- $[x]$ is finite and $[x] = P \cap \Gamma$.

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- Cycle relations (This should be cyclic):
  - Let $S_1 = S$ for a given side S. Choose the side R of S_1. Obtain S_1'. Let S_2 be the side adjacent to S_1' so that $g_{S_1}(S_1' \cap S_2) = R$.
  - Let S_{i+1} be the side of P adjacent to S_i' such that $g_{S_i}(S_i' \cap S_{i+1}) = S_{i-1}' \cap S_i$.
- Then
  - There is an integer I such that $S_{i+I} = S_i$ for each i.
  - $\sum_{i=1}^{I} \theta(S_i', S_{i+1}) = 2\pi/k$.
  - $g_{S_1} g_{S_2} \ldots g_{S_I}$ has order k.
- Example: the octahedron in the hyperbolic plane giving genus 2-surface.
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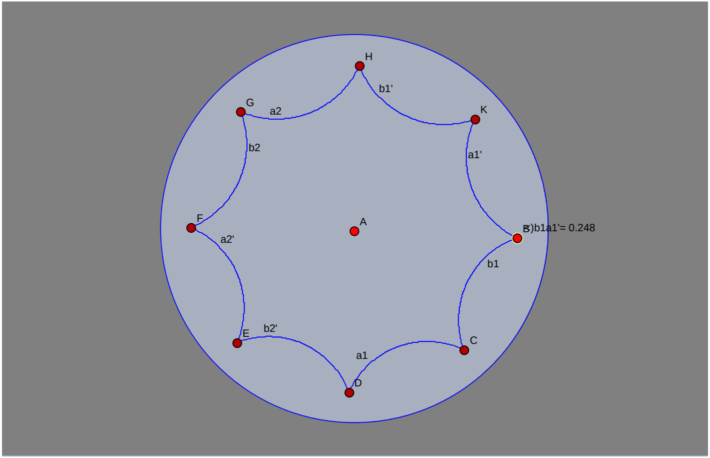

- $(a1, D), (a1', K), (b1', K), (b1, B), (a1', B), (a1, C), (b1, C),$
- $(b1', H), (a2, H), (a2', E), (b2', E), (b2, F), (a2', F), (a2, G),$
- $(b2, G), (b2', D), (a1, D), (a1', K), \ldots$

- Poincare fundamental polyhedron theorem is the converse. (See Kapovich P. 80–84):
- Given a convex polyhedron P in X with side-pairing isometries satisfying the above relations, then P is the fundamental domain for the discrete group generated by the side-pairing isometries.
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Reflection groups

- A discrete reflection group is a discrete subgroup in G generated by reflections in X about sides of a convex polyhedron. Then all the dihedral angles are submultiples of π.
- Then the side pairing such that each face is glued to itself by a reflection satisfies the Poincare fundamental theorem.
- The reflection group has presentation $\{S_i : (S_iS_j)^{k_{ij}}\}$ where $k_{ii} = 1$ and $k_{ij} = k_{ji}$.
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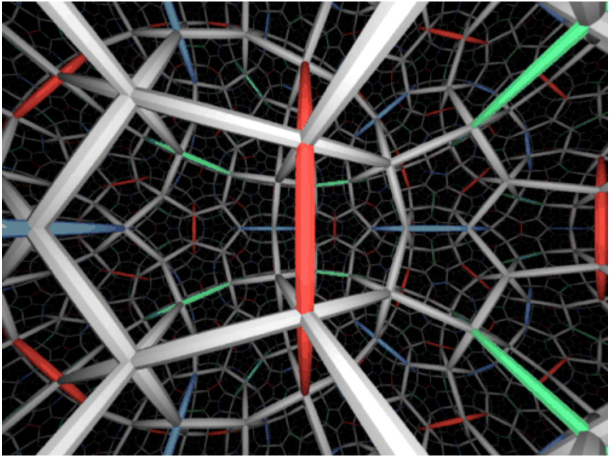
Dodecahedral reflection group

One has a regular dodecahedron with all edge angles $\pi/2$ and hence it is a fundamental domain of a hyperbolic reflection group.



Triangle groups

- Find a triangle in X with angles submultiples of π.
- We divide into three cases $\pi/a + \pi/b + \pi/c > \pi, = \pi, < \pi$.
- We can always find ones for any integers a, b, c.
  - $> \pi$ cases: $(2,2,c), (2,3,3), (2,3,4), (2,3,5)$ corresponding to dihedral group of order $4c$, a tetrahedral group, octahedral group, and dodecahedral group.
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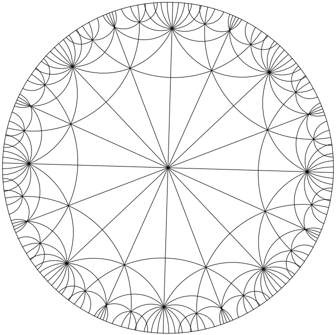
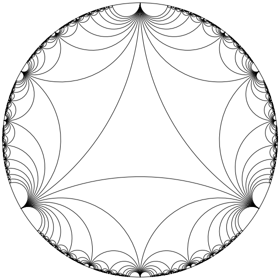
- (2, 4, 8)-triangle group



- The ideal example
  `http://egl.math.umd.edu/software.html`



Higher-dimensional examples

- To construct a 3-dimensional examples, obtain a Euclidean regular polytopes and expand it until we achieve that all angles are $\pi/3$. Regular octahedron with angles $\pi/2$. These are ideal polytope examples.
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- Bieberbach theorem:
  - A group is isomorphic to a crystallographic group if and only if it contains a subgroup of finite index that is free abelian of rank equal to the dimension.
  - The crystallographic groups are isomorphic as abstract groups if and only if they are conjugate by an affine transformation.

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Crystallographic groups

- There are only finitely many crystallographic group for each dimension since once the abelian group action is determined, its symmetry group can only be finitely many.
- 17 wallpaper groups for dimension 2. http://www.clarku.edu/~djoyce/wallpaper/ and see Kali by Weeks http://www.geometrygames.org/Kali/index.html.
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