An introduction to convex real projective manifolds (orbifolds)

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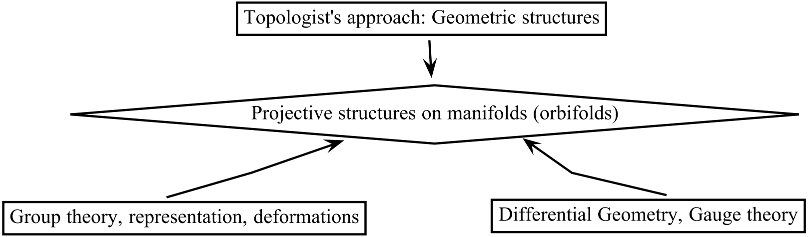
Abstract

- A projectively flat manifold (orbifold) is a manifold (orbifold) with an atlas of charts to the projective space with transition maps in the projective automorphism group. These objects are closely related to the representations of groups into the projective groups $PGL(n+1, \mathbb{R})$.
- We will give a partial survey of this area including the classical results and many recent results by Goldman, Loftin, Labourie, Benoist, Cooper, Long, and so on using many diverse methods from low-dimensional topology, group representations, and affine differential geometry.



Topologist's approach: Geometric structures
Projective structures on manifolds (orbifolds)
Group theory, representation, deformations
Differential Geometry, Gauge theory

What are the objectives?

- What kind of geometric (projective) structures are on a given manifold (orbifold)?
- Determine the topology and geometry of the deformation space

$$D(M) = \{\text{geometric (projective) structures on } M\}/\text{isotopies}$$

in relation to the objects

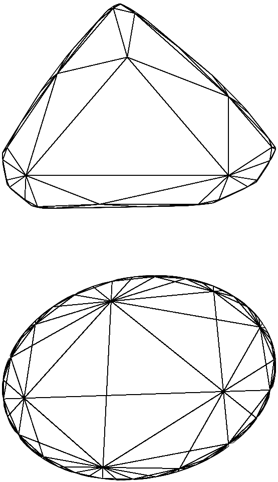
$$\mathrm{Hom}(\pi_1(M), \mathrm{PGL}(n+1, \mathbb{R}))/\sim \text{ or } \mathrm{Hom}(\pi_1(M), \mathrm{SL}_{\pm}(n+1, \mathbb{R}))/\sim$$

- How do the geometry and topology and algebra interact from the perspective of manifold and orbifold theory?
- After any rigid type geometric structures having been studied much, we should consider flexible types ones: conformally flat, projectively flat, or affinely flat geometric structures.

Origins in Geometry

- Cartan defined projectively flat structures on manifolds as:
  - "geodesically Euclidean but with no metrics"
  - torsion-free
  - projectively flat (i.e., same geodesics structures as flat metrics)
  - affine connection on manifolds.
- Chern worked on projective differential geometry.
- Ehresmann identifies this structure as having a maximal atlas of charts
  - to $\mathbb{RP}^n$
  - with transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$.
- Kuiper on closed convex projective surfaces Ω/Γ of negative Euler characteristic:
  - $\partial\Omega$ is strictly convex,
  - Every closed curve is realized as a closed projective geodesic.

- Most examples of projective manifolds are by taking quotients of a domain Ω in $\mathbb{R}P^n$ by a discrete subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$.
- The domains are usually convex and we call the quotient *convex projective manifold (or orbifolds)*. There are of course projective manifolds that are not from domains.
- Let H^n be the interior of an ellipsoid. Then H^n is the hyperbolic space and $Aut(H^n)$ is the isometry group. H^n/Γ has a canonical projective structure.
- Benzécri made some extensive study of these from the point of view of convex bodies.
- J.L. Koszul showed that the convexity is preserved if one slightly changed the projective structures.
- E. Vinberg studied reflection groups acting on properly convex cones. He classified using the Cartan matrices. These include hyperbolic reflection groups.
- Kac-Vinberg were first to find examples of convex projective surfaces that are not hyperbolic. The examples are based on Coxeter groups. (This group studies this subject by the linear actions on convex cones in $\mathbb{R}^n$.)
- See Goldman's lecture notes for an exposition.



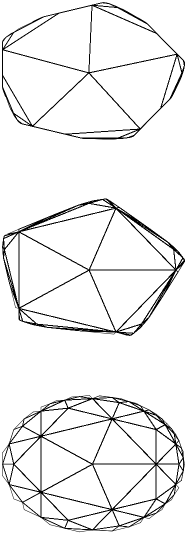


Figure: Orbifolds based on 2-spheres with three cone points. The files can be found in my homepages.

Kobayashi on convex projective manifolds

He considers maps

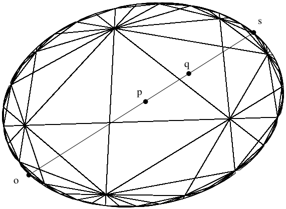
$$I \subset \mathbb{RP}^1 \to M$$

and take maximal ones. (I proper intervals or a complete real line. This defines a pseudo-metric.

- Kobayashi metric is a metric if and only if
  - there are no complete affine lines if and only if
  - $M = \Omega/\Gamma$ where Ω is a properly convex domain in $\mathbb{RP}^n$.
- In this case, Kobayashi metric is Finsler and equals a Hilbert metric

$$d_\Omega(p, q) = \log(o, s, q, p).$$

If $\Omega = H^n$, the metric is the standard hyperbolic metric.



Affine differential geometry and projective manifolds

- A projective structure on manifold M induces an affine structure on $M \times \mathbb{R}$ and conversely:
  - Given a convex domain Ω in $\mathbb{RP}^n$, we can obtain a convex cone V in $\mathbb{R}^{n+1}$ by taking only the positive rays corresponding to Ω.
  - Conversely, given a convex cone V in $\mathbb{R}^{n+1}$, we obtain a convex domain Ω in $\mathbb{RP}^n$.
- An affine sphere is a complete hypersurface asymptotic to Ω and an affine normal line passing through the origin.
- An affine sphere structure on a closed manifold M means that $\tilde{M}$ imbeds as an affine sphere asymptotic to the cone V corresponding to the convex domain Ω where the deck transformation group acts as a discrete linear group.

Cheng-Yau theory

- John Loftin (simultaneously Labourie near 1994) showed using Calabi, Cheng-Yau's work on affine spheres: Let M^n be a closed projectively flat manifold.
- Then M is properly convex if and only if M admits an affine sphere structure based in $\mathbb{R}^{n+1}$.
- C.P. Wang showed in 70s that in dimension 2: A conformal structure on an oriented surface S of genus 2 and a holomorphic section of K^3 determine an affine sphere structure on S.
- In particular, this shows that the deformation space $D(\Sigma)$ of convex projective structures on Σ admits a complex structure, which is preserved under the moduli group actions.
- Loftin also worked out Mumford type compactifications of the moduli space $M(\Sigma)$ of convex projective structures.
- He has found new metrics on $D(\Sigma)$ based on harmonic maps. (There is a Weil-Peterson type pressure metric developed by M. Bridgeman and D. Canary recently. Also, Goldman's one using Vinberg functions.)

Gauge theory and projective structures

- Atiyah and Hitchin studied self-dual connections on surfaces (70s)
- Corlette showed that flat connections for manifolds (80s) can be realized as harmonic maps to certain symmetric space bundles.
- A Teichmüller space

$$T(\Sigma) = \{\text{hyperbolic structures on } \Sigma\}/\text{isotopies}$$

  is a component of

$$Hom^{+}(\pi_1(\Sigma), PSL(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

  of fathful discrete Fuchian representations.
- Hitchin-Teichmüller component:

$$\Gamma \overset{Fuchsian}{\rightarrow} PSL(2,\mathbb{R}) \overset{irreducible}{\rightarrow} G. \tag{1}$$

  gives a component of

$$Hom^{+}(\pi_1(\Sigma), G)/G.$$

Hitchin-Teichmüller components

Gauge theory for flat bundles

Hitchin used Higgs field on principal G-bundles over surfaces to obtain parametrizations of flat G-connections over surfaces. (G is a real split form of a reductive group.) (90s)

- A Higgs bundle is a pair (V, Φ) where V is a holomorphic vector bundle over Σ and Φ is a holomorphic section of $EndV \otimes K$.
- To find a flat connection given a Higgs bundle, we solve for A

$$F_A + [\Phi, \Phi^*] = 0.$$

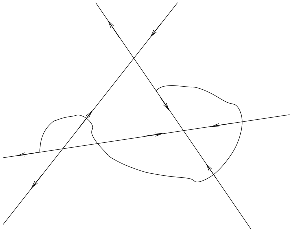
- The Hitchin-Teichmüler component is homeomorphic to a cell of dimension $(2g - 2)\dim G^r$.
- For $n > 2$,

$$Hom^+(\pi_1(\Sigma), PGL(n, \mathbb{R}))/PGL(n, \mathbb{R})$$

has three connected components if n is odd and six components if n is even.

Topologist's approach

- Topologist's approach is to study more general structures with developing maps that are immersions.
- Benzecri (and Milnor) showed that an affinely flat 2-manifold has Euler characteristic $= 0$ (Chern conjecture).
- Benzecri studied convex domains Ω that arise for convex projective manifolds Ω/Γ. The boundary of Ω is C^1 or Ω is an ellipsoid. (1960)
- Nagano and Yagi classified affine structures on tori. (1976)



- Goldman classified projective structures on tori. (His senior thesis)
- Grafting: One can insert this type of annuli into a convex projective surfaces to obtain non-convex projective surfaces.

Theorem 1 (Convex decomposition (1994))

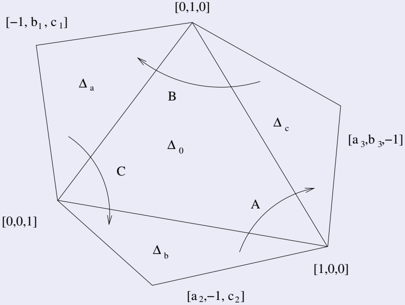
Given a closed orientable real projective surface Σ of negative Euler characteristic, Σ has a disjoint collection of simple closed curves decomposing it to a union of properly convex projective surface and annuli.

Goldman's classification of convex projective structures on closed surfaces (1990): Determining the deformation space $D(\Sigma)$, $\chi(\Sigma) < 0$:

- first cut up the surface into pairs of pants.
- Each pair of pants is a union of two open triangles.
- We realize the triangles as geodesic ones.
- We investigate the projective invariants of union of four triangles in $\mathbb{RP}^2$.
- The needed key is that

$$D(P) \to D(\partial P)$$

  for a pair of pants P is a principle fibration for a pair of pants P.
- There is also $\mathbb{R}^2$ ways to glue a pair of s.c. geodeiscs.
- Hence, $D(\Sigma)$ is homeomorphic to $\mathbb{R}^{-8\chi(\Sigma)}$.



The deformation space continued

- We can generalize this to 2-orbifolds.

Theorem 2 (Choi-Goldman)

Let Σ be a closed orbifold of orbifold Euler characteristic < 0. Then the deformation space $D(\Sigma)$ of convex real projective structures on Σ is homeomorphic to a cell of dimension

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where

- *X_Σ is the underlying space,*
- *k_c the number of cone-points, b_c the number of cone-points of order 2*
- *k_r the number of corner reflectors, and b_r the number of corner-reflectors of order 2.*

Hitchin's conjecture and the generalizations

- A convex projective surface is of form Ω/Γ. Hence, there is a representation $\pi_1(\Sigma) \to \Gamma$ determined only up to conjugation by $\mathrm{PGL}(3, \mathbb{R})$. This gives us a map

  $$hol : D(\Sigma) \to Hom(\pi_1(\Sigma), \mathrm{PGL}(3, \mathbb{R}))/\sim .$$

  This map is known to be a local-homeomorphism (Ehresmann, Thurston) and is injective (Goldman)
- The map is in fact a homeomorphism onto the Hitchin-Teichmüller component (—,Goldman) The main idea for proof is to show that the image of the map is closed. (For $n = 2$, this is a classical theorem due to Weil.)

Labourie generalized this to $n \geq 2, 3$ (2006,2007)

- A Hitchin representation in $PSL(n, \mathbb{R})$: a representation deformable to a Fuchsian representation. i.e., those in the Hitchin-Teichmüller component.

Theorem 3

If ρ is a Hitchin representation, then there exists a (unique) ρ-equivariant hyperconvex curve ζ, the limit curve, from $\partial_\infty \Gamma \to \mathbb{R}P^{n-1}$.

- A continuous curve $\eta : S^1 \to \mathbb{R}P^{n-1}$ is *hyperconvex* if for any distinct points $(x_1, \ldots, x_n)$ in S^1,

$$\eta(x_1) + \cdots + \eta(x_n)$$

  is a direct sum.

- The proof uses Anosov type dynamical geometrical structures and stability argument.

Corollary 4 (Discrete and faithful component)

Every Hitchin representation is a discrete faithful and "purely loxodromic". The mapping class group acts properly on $H(n)$.

Group theory and representations: Results of Benoist (2003-2006)

- As stated earlier, Kac-Vinberg, Koszul started to study the deformations of representations $\Gamma \rightarrow \mathrm{PGL}(n+1, \mathbb{R})$ dividing a properly convex domain in $\mathbb{R}P^n$. We wish to know the space of characters.
- There is a well-known deformation called "bending" for projective and conformally flat structures.
- Johnson and Millson found that for certain hyperbolic manifold has a deformation space of projective structures that is singular. (They also worked out this for conformally flat structures.)

Benoist completed this theory in a sense (papers "Convex divisibles I-IV"):

Theorem 5

Γ an irreducible torsion-free subgroup of $\mathrm{PGL}(m, \mathbb{R})$. Then Γ acts on a strictly convex domain Ω if and only if Γ is positive proximal. If Γ acts divisibly on strictly convex Ω, and Ω is not the interior of an ellipsoid, then Γ is Zariski dense in $\mathrm{PGL}(m, \mathbb{R})$.

Theorem 6

Let Γ be a discrete torsion-free subgroup of $\mathrm{PGL}(m, \mathbb{R})$ dividing an open strictly convex domain in $\mathbb{R}P^{m-1}$. Let C be the corresponding cone on $\mathbb{R}^m$. Then one of the following holds.

- *C is a product, i.e., reducible.*
- *C is homogeneous, or*
- *Γ is Zariski dense in $\mathrm{PGL}(m, \mathbb{R})$.*

- If the virtual center of Γ_0 is trivial, then

$$E_{\Gamma_0} = \{\rho \in H_{\Gamma_0} | \text{ The image of } \rho \text{ divides a convex open domain in } \mathbb{R}P^{n-1}.\}$$

is closed in

$$H_{\Gamma_0} := Hom(\Gamma_0, PGL(m, \mathbb{R}))$$

.

Openness was obtained by Koszul.

- Let Γ be as above. Then the following conditions are equivalent:
  - Ω is strictly convex.
  - $\partial\Omega$ is C^1.
  - Γ is Gromov-hyperbolic.
  - Geodesic flow on Ω/Γ is Anosov.

Search for deformations in higher-dimensional real projective manifolds or orbifolds

Hyperbolic orbifolds:

The hyperbolic 3-manifold (orbifold) determines a discrete faithful representation of its fundamental group into $PSL(2, \mathbb{C})$, or equivalently into $SO^+(3, 1)$. This representation is unique up to conjugation by the Mostow rigidity.

(Cooper, Long, Thistlethwaite 2006, 2007)

If we consider $G := SO^+(3, 1)$ as a subgroup of a larger group $\hat{G}$, we can search for deformations of the G-representation into the group $\hat{G}$.

- Out of the first 5000 closed hyperbolic 3-manifolds in the Hodgson-Weeks census, a handful (5%) admit non-trivial deformations of their $SO^+(3, 1)$-representations into $SL(4, \mathbb{R})$;
- each resulting representation variety then gives rise to a family of convex projective structures on the manifold.

Dehn surgery results

Heusener-Porti (2011) showed the projective rigidity of infinitely many 3-manifolds obtained by Dehn-surgeries from an infinitesimally projectively rigid cusped hyperbolic manifolds.

- In particular applies to the hyperbolic complements of tunnel number one knots. Some recent extension by Ballas.
- Question: Calculate the dimension of real projective structures on knot complements and relate the result with the results of CLT.

Deformation spaces of Coxeter orbifolds

- A *Coxeter n-orbifold* $\hat{P}$ is an n-dimensional orbifold based on a polytope P with silvered boundary facets.

The *deformation space* $D(\hat{P})$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.

- We follow Vinberg's analysis in 70s. Benoist also made further investigations.
- A point p of $D(\hat{P})$ always determines a fundamental polyhedron P up to projective automorphisms. We wish to understand the space where the fundamental polyhedron is always projectively equivalent to P.

This is the *restricted deformation space* of $\hat{P}$ and we denote it by $D_P(\hat{P})$.

Orderable Coxeter 3-orbifolds

Definition 5.1

Let $\hat{P}$ be the orbifold obtained from P by silvering sides and removing vertices as above. $\hat{P}$ is *orderable* if we can order the sides of P so that each side has no more than three edges which are either of order 2 or included in a side of higher index.

Theorem 5.2 (2006)

Let P be a convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P) = \dim \mathbf{Aut}(P)$. Suppose that $\hat{P}$ is orderable. Then $D_P(\hat{P})$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty. (*e the number of edge, e_2 the number of edges of order 2.*)

Proof.

The basic idea is to control the reflection points in order. Basically, this is the "underdetermined case" in terms of algebraic equations. (Others are usually "overdetermined cases".) □

The total deformation space fibers over the open subspace of polytopes combinatorially equivalent to P.

Iterated-truncation tetrahedron (ecimaedre combinatoire)

- We start with a tetrahedron and cut off a vertex. We iterate this. This gives us a convex polytope with trivalent vertices, i.e., truncation polytope.

Theorem of L. Marquis (2010)

$D(\hat{P})$ of a compact hyperbolic Coxeter 3-orbifold $\hat{P}$ based on a truncation polytope is diffeomorphic to $\mathbb{R}^{e-e_2-3}$.

- The proof is basically very combinatorial and algebraic over $\mathbb{R}$. (generalizations?)
- (Choudhury, Lee, Choi) Orderable compact hyperbolic Coxeter orbifolds are only five types: tetrahedron, prism, and three other. There are infinitely many orderable noncompact Coxeter 3-orbifolds admitting hyperbolic structures.
- The orderbility is more general then truncation orbifold conditions; however, for compact ones, they are the same.

Hyperbolic ideal (or hyperideal) Coxeter 3-orbifolds

Theorem 7 (Lee, Hodgson, Choi)

For ideal or hyperideal hyperbolic Coxeter 3*-orbifold* $\hat{P}$ *with all edge orders* ≥ 3, $D_P(\hat{P})$ *is locally a smooth cell of dimension* 6 *at the hyperbolic point, and* $D(\hat{P})$ *has dimension* $e - 3$ *and smooth at the hyperbolic point.*

Proof.

The proof involves the Weil-Prasad infinitesimal rigidity: □

Numerical experiments on cubes and dodecahedrons

Following up on the Cooper-Long-Thistlethwaite approach,

Theorem 8 (Choi, Hodgson, Lee (2012))

Consider the compact hyperbolic cubes such that each dihedral angle is $\pi/2$ or $\pi/3$. Up to symmetries, there exist 34 cubes satisfying this condition. For the corresponding hyperbolic Coxeter orbifolds,

- *10 are projectively deformable relative to the mirrors*
- *and the remaining 24 are projectively rigid relative to the mirrors.*
- *The deformations of* 3 *orbifolds are not projective bendings.*

Some of these with many 2s are shown to be rigid by "a linear test". We use computations packages and some of these need Gröbner basis techniques.

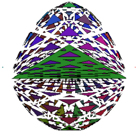
Projective deformations of weakly orderable hyperbolic Coxeter orbifolds (with Gye-Seon Lee, 2012)

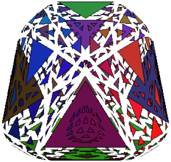
- A compact Coxeter orbifold Q is *weakly orderable* if we can order the faces of Q so that each face contains at most 3 edges of order 2 in faces of higher indices, or Q is based on a truncation polytope.
- Many more compact hyperbolic Coxeter orbifolds satisfy this condition. (Given a polytope satisfying certain conditions, with probabilities limit to 1.)

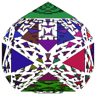
Theorem 9

Let Q be a compact weakly orderable Coxeter orbifold with a hyperbolic structure. Then $D(Q)$ is a smooth manifold of dimension $e - e_2 - n$ at the hyperbolic point if $n = 3$ and Q is weakly orderable or Q is based on a truncation polytope.

More pictures (due to Yves Benoist)







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