Compactifications of Margulis space-times

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Abstract

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- The compactification of Margulis space-times by attaching closed $\mathbf{R}P^2$-surfaces at infinity (when the groups do not contain parabolics.) The compactified spaces are homeomorphic to solid handlebodies.
- Finally, we will discuss about the parabolic regions of tame Margulis space-times with parabolic holonomies.
- There is another contemporary approach by Danciger, Gueritaud and Kassel.

Content

Background

- Flat Lorentz space $\mathsf{E} = \mathbb{R}^{2,1}$ is $\mathbb{R}^3$ with $Q(x, y, z) = x^2 + y^2 - z^2$.
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- A free group Γ of rank ≥ 2 acting properly on E and freely. $\Gamma \cong F_n$.
- E/Γ is called a *Margulis space-time*.
- $L(\Gamma) \subset \mathrm{SO}(2,1)$. Assume $L(\Gamma) \subset \mathrm{SO}(2,1)^o$ and that this is a Fuchsian group. (It must be free by G. Mess)
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- An element g of Γ is of form $g(x) = L(g)x + b_g$ for $L(g) \in SO(2, 1)$ and $b_g \in \mathbb{R}^{2,1}$.
- Γ is classified by $[b] \in H^1(F_n, \mathbb{R}^{2,1}_{L(\Gamma)})$.
- Γ is called a proper affine deformation of $L(\Gamma)$, and are classified by Goldman, Labourie, and Margulis [7].
- The topology of E/Γ is in question here.

The tameness

- $L(\Gamma)$ is convex cocompact if it has a compact convex hull. That is it does not contain a parabolic.

Theorem 1.1 (Goldman-__ , Danciger-Gueritaud-Kassel)

Let $\mathbb{R}^{2,1}/\Gamma$ be a Margulis spacetime. Assume Γ has no parabolics. Then $\mathbb{R}^{2,1}/\Gamma$ is homeomorphic to a handlebody of genus n.

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- This follows from [3]:

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- DGK also proved the crooked plane conjecture. The tameness and the compactification follow from this result also.

The real projective geometry

- $\mathbb{R}P^n = P(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} - \{O\}/\sim$ where $v \sim w$ if $v = sw$ for $s \neq 0$.
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- The oriented version $\mathbf{S}^n := S(\mathbb{R}^{n+1}) = \mathbb{R}^{n+1} - \{O\}/\sim$ where $v \sim w$ if $v = sw$ for $s > 0$.
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- The projection $(x_1, \cdots, x_{n+1}) \rightarrow ((x_1, \cdots, x_{n+1}))$, the equivalence class.

Affine geometry as a sub-geometry of projective geometry

- $\mathbb{R}^n = H^o \subset H \subset \mathbf{S}^n$ where H is a hemisphere.
- $\mathrm{Aff}(\mathbb{R}^n) = Aut(H)$

$$= \left\{ \begin{pmatrix} A & b \\ 0 & \lambda \end{pmatrix} \middle| A \in \mathrm{GL}(n, \mathbb{R}), b \in \mathbb{R}^n, \lambda > 0. \right\}$$

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- $H^n = \mathbb{R}^n \cup \mathbf{S}^{n-1}_\infty$ for a hemisphere H^n of $\mathbf{S}^n$ with the ideal boundary $\mathbf{S}^{n-1}_\infty$.
- A complete affine manifold is of form $H^{n,o}/\Gamma$ for $\Gamma \subset \mathrm{Aut}(H^n)$ and the group $\mathrm{Aut}(H^n)$ of projective automorphism of H^n, equal to $\mathrm{Aff}(H^{n,o})$.

Lorentz geometry compactified

- $\mathbb{R}^{2,1} = \mathscr{H}^o$ the interior of a 3-hemisphere $\mathscr{H}$ in $\mathbf{S}^3$.
- $\mathrm{Isom}(\mathbb{R}^{2,1}) = \mathbb{R}^3 \rtimes \mathrm{SO}(2,1) \hookrightarrow \mathrm{Aut}(\mathscr{H}) \hookrightarrow \mathrm{SL}_{\pm}(4,\mathbb{R})$.

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- $\mathscr{H} = \mathbb{R}^{2,1} \cup \mathbf{S}^2_\infty$ is the compactification of $\mathbb{R}^{2,1}$ with the ideal boundary $\mathbf{S}^2_\infty$.
- A element of $\mathrm{Isom}(\mathbb{R}^{2,1})$ with a semisimple linear part is a Lorentzian boost.

$$g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{pmatrix} e^l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-l} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}.$$

Lorentzian boost

- As an element of $\mathrm{SL}_{\pm}(4,\mathbb{R})$,

$$\gamma = \begin{pmatrix} e^{l} & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha \\ 0 & 0 & e^{-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \alpha \neq 0.$$

The six fixed points on $\mathbf{S}^2_\infty$ are:

$$x^{+}_{\pm} :=((\pm 1 : 0 : 0 : 0)),\ x^{0}_{\pm} :=((0 : \pm 1 : 0 : 0)),\ x^{-}_{\pm} :=((0 : 0 : \pm 1 : 0)).$$

in homogeneous coordinates on $\mathbf{S}^2_\infty$.

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- $\mathrm{Axis}(\gamma) := \overline{x^0_+ O x^0_-}$. γ acts as a translation on $\mathrm{Axis}(\gamma)$ towards x^0_+ when $\alpha > 0$ and towards x^0_- when $\alpha < 0$.
- Define the weak stable plane $\mathscr{W}^{wu}(\gamma) := \mathrm{span}\left(x^-(\gamma) \cup \mathrm{Axis}(\gamma)\right)$.

Projective boosts

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- A *Lorentzian boost* is any isometry g conjugate to γ.
- A *projective boost* is a projective extension $\mathbf{S}^3$ of a Lorentzian boost.
- The elements $x_\pm^\pm(g), x_\pm^0(g)$ are all determined by g itself.

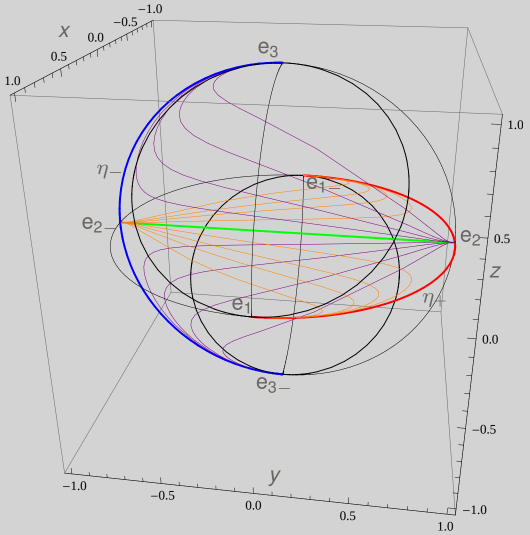

Figure: The action of a projective boost $\hat{g}$ on the 3-hemisphere $\mathscr{H}$ with the boundary sphere $\mathbf{S}^2_\infty$.

Convergence for projective boosts

- a projective automorphism $g_{\lambda,k}$ of form

$$\begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & k \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \lambda > 1, k \neq 0 \tag{1}$$

under a homogeneous coordinate system of $\mathbf{S}^3$ is a projective-boost.

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  under a homogeneous coordinate system of $\mathbf{S}^3$ is a projective-boost.
- Assume $(\lambda, k) \to \infty$ and $k/\lambda \to 0$.
  (a) $g_{\lambda,k}|\mathscr{H} - \mathbf{S}^2_\infty$ attracting fixed points e_1, e_{1-}.
  (b) $g_{\lambda,k}|(S \cap \mathscr{H}) - \eta_- \to e_2$ for the stable subspace S.
  (c) $K \subset \mathscr{H} - \eta_-$, K meets both component of $\mathscr{H} - S$. Then $g_{\lambda,k}(K) \to \eta_+$.

The ideal boundary $\mathbf{S}^2_\infty$ of E

- The *sphere of directions* $\mathbf{S}^2_\infty := \mathbb{S}(\mathbb{R}^{2,1})$ double-covering $\mathbb{R}\mathrm{P}^2$.
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- The image $\mathbb{S}_+$ of the space of future timelike vectors identifies with the hyperbolic 2-plane $\mathbb{H}^2$, (the Beltrami-Klein model of the hyperbolic plane.)
- Let $\mathbb{S}_-$ denote the subspace of $\mathbb{S}$ corresponding to past timelike vectors.
- $SO(2,1)^o$ acts faithfully on $\mathbb{H}^2 = \mathbb{S}_+$ as the orientation-preserving isometry group and $SO(2,1)$ acts so on $\mathbb{S}_+ \cup \mathbb{S}_-$ and acts on $\mathbf{S}^2_\infty$ projectively.
- Let $\mathbb{S}_0 := \mathbf{S}^2_\infty - \mathrm{Cl}(\mathbb{S}_+) - \mathrm{Cl}(\mathbb{S}_-)$.

Oriented Lorentzian space E

- $\mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \to \mathbb{R}, (v, u) \mapsto v \cdot u.$
- $\mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \to \mathbb{R}, (v, u, w) \mapsto Det(v, u, w).$

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- Null space $\mathcal{N} := \{v | v \cdot v = 0\} \subset \mathbb{R}^{2,1}$.
- Let $v \in \mathcal{N}$, $v \neq 0$, $v^{\perp} - \mathbb{R}v$ has two choices of components.
- Define the *null half-plane* $\mathscr{W}(\mathrm{v})$ (or the *wing*) associated to v as:

$$\mathscr{W}(\mathrm{v}) := \left\{\mathrm{w} \in \mathrm{v}^{\perp} \mid \mathrm{Det}(\mathrm{v}, \mathrm{w}, \mathrm{u}) > 0\right\} \subset \mathrm{v}^{\perp} - \mathbb{R}\mathrm{v}.$$

  where u is chosen arbitrarily in the same $\mathrm{Cl}(\mathbb{S}_{\pm})$ that v is in.
- $\mathscr{W}(\mathrm{v}) = \mathscr{W}(-\mathrm{v})$.

- The corresponding set of directions is the open arc

$$\varepsilon(\!((\mathsf{v}))\!) := ((\mathscr{W}(\mathsf{v})))$$

in $\mathbb{S}_0$ joining $((\mathsf{v}))$ to its antipode $((\mathsf{v}_-))$.

- The corresponding map

$$((\mathsf{v})) \longmapsto \varepsilon(\!((\mathsf{v}))\!)$$

is an $\mathsf{SO}(2,1)$-equivariant map

$$\partial\mathbb{S}_+ \to \mathcal{S}$$

where $\mathcal{S}$ denotes the set of half-arcs.

Figure: The tangent geodesics to disks $\mathbb{S}_+$ and $\mathbb{S}_-$ in the unit sphere $\mathbf{S}^2_\infty$ imbedded in $\mathbb{R}^3$.

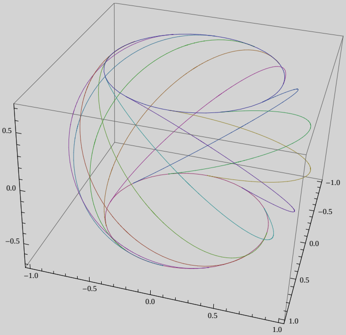
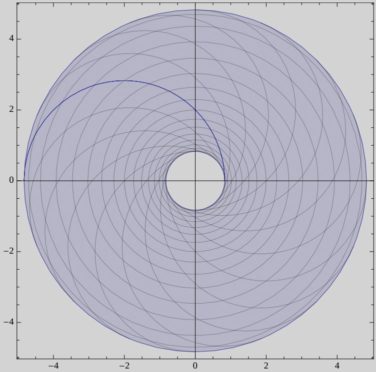


Figure: The tangent geodesics to disks $\mathbb{S}_+$ and $\mathbb{S}_-$ in the stereographically projected $\mathbf{S}^2_\infty$ from $(0, 0, -1)$. The inner circle represents the boundary of $\mathbb{S}_+$. The arcs of form $\varepsilon(x)$ for $x \in \partial\mathbb{S}_+$ are leaves of the foliation $\mathcal{F}$ on $\mathbb{S}_0$.



$\mathbb{R}P^2$-surfaces to bordify E/Γ.

- $\Sigma_+ := \mathbb{S}_+/L(\Gamma)$ is a complete hyperbolic surface without parabolics.
- We can add finitely many arcs to compactify $\Sigma'_+ := \Sigma_+ \cup c_1 \cup \cdots \cup c_n$.
- $\tilde{\Sigma}'_+ = \mathbb{S}_+ \cup \bigcup_{i \in \mathcal{J}} \mathbf{b}_i = \mathrm{Cl}(\mathbb{S}_+) - \Lambda$.

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- We define

$$\begin{aligned} \tilde{\Sigma} &= \tilde{\Sigma}'_+ \cup \coprod_{i \in \mathcal{J}} \mathcal{A}_i \cup \tilde{\Sigma}'_- \\ &= \mathbf{S}^2_\infty - \bigcup_{x \in \Lambda} \mathrm{Cl}(\varepsilon(x)). \end{aligned} \tag{2}$$

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Theorem 3.1 (Projective Schottky surface)

$L(\Gamma)$ acts properly discontinuously and freely on $\tilde{\Sigma}$, and $\Sigma := \tilde{\Sigma}/L(\Gamma)$ is a closed $\mathbb{R}P^2$-surface. The same is true for $\mathscr{A}(\tilde{\Sigma})$.

The proper action of Γ on $E \cup \tilde{\Sigma}$

- Recall the Margulis invariant:

$$\mu(g) = B(gx - x, v(g))$$

where $v(g)$ is the unit space-like neutral vector of g

$$v(g) := \frac{x^+(g) \times x^-(g)}{||x^+(g) \times x^-(g)||}.$$

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- To obtain the converse, the diffused Margulis invariants are introduced by Goldman, Labourie, and Margulis [7]. We will use their techniques.
- We know that Γ acts properly on E and $\tilde{\Sigma}$ separately.

- Let $C(\Sigma_+)$ be the space of Borel probability measures on $\mathbb{U}\Sigma_+/\Gamma$ invariant under the flow. These are supported on the nonwondering part $\mathbb{U}_{rec}\Sigma_+$.
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Theorem 4.1 (Goldman-Labourie-Margulis)

μ has the same sign if and only if Γ acts properly on $\mathbb{R}^{2,1}$.

A consequence of the proof:

There exists a continuous section

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- Goldman-Labourie found a one to one correspondence

$$\{l | l \text{ is a nonwandering geodesic on } \Sigma_+\} \leftrightarrow \{l | l \text{ is a nonwandering spacelike geodesic on } E/\Gamma\}$$

- **A key fact:** Closed geodesics on Σ_+ corresponds to closed geodesic in E/Γ. The set is precompact

The proof of the properness

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- Let a_i and r_i on $\partial \mathbb{S}_+$ denote the attracting and the repelling fixed points of g_i.
- Since a Fuchsian group is a convergence group: There exists a subsequence g_i so that

$$a_i \to a, b_i \to b, a, b \in \partial \mathbb{S}_+, \ \lambda_i \to \infty.$$

- Assume $a \neq b$. The other case will be done by "Margulis trick"

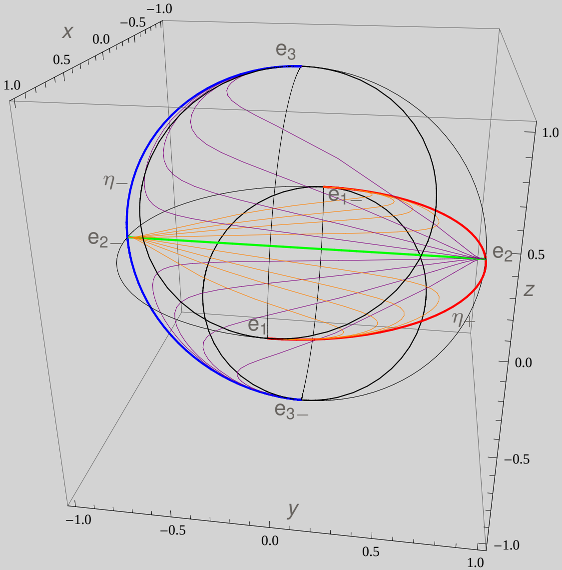

Figure: The actions are very close to this one.

- By the boundedness of nonwandering spacelike geodesics in E/Γ, we find a bounded set $h_i \in \mathrm{SL}_\pm(4, \mathbb{R})$ so that $h_i g_i h_i^{-1}$ is in the standard form.
- We choose a subsequence $h_i \to h_\infty \in \mathrm{SL}_\pm(4, \mathbb{R})$ and the stable subspace $S_i \to S_\infty$.
- Cover K by a finite set of convex ball meeting S_∞ and ones disjoint from S_∞.
- $g_i(B) \to a, a_-$ for disjoint balls.
- $g_i(B') \to \varepsilon(a)$ for B' meeting S_∞.

Compactness

- Now we know $E \cap \tilde{\Sigma}/\Gamma$ is a manifold.
- We know Σ is a closed surface. $\pi_1(\Sigma) \to \pi_1(M)$ surjective.

Margulis spacetime with parabolics

- In analogy with the thick and thin decomposition of hyperbolic 3-manifolds.
- Let $g \in \mathrm{Isom}(\mathbb{R}^{2,1})$ with $L(g)$ parabolic. Suppose that g acts properly on E.
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Theorem 6.1 (Charette-Drumm)

If Γ acts properly on E, $\{\mu^{\mathrm{gen}}(g) | g \in \Gamma\}$ *have the same signs.*

- The converse is being proved by Goldman, Labourie, Margulis, Minsky [8].

Understanding the parabolic transformation

- We restrict to the cyclic $\langle g \rangle$ for parabolic g.
- $U = \exp(N) = \mathrm{I} + N + \frac{1}{2}N^2$ where N is skew-adjoint nilpotent.
- $N = \log(U) = (U - \mathrm{I}) + \frac{1}{2}(U - \mathrm{I})^2$.

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Lemma 6.2 (Skew-Nilpotent)

There exists $c \in KerN$ such that c is causal, $c = N(b)$, $b \in \mathbb{R}^{2,1}$ is space-like and $b.b = 1$.

- Using basis $\{a, b, c\}$ with $b = N(a), c = N(b)$, we obtain a one-parameter family containing U

$$\Phi(t) : \mathsf{E} \to \mathsf{E} = \begin{pmatrix} 1 & t & t^2/2 & \mu t^3/6 \\ 0 & 1 & t & \mu t^2/2 \\ 0 & 0 & 1 & \mu t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $\phi = y\partial_x + z\partial_y + \mu\partial_z$ is the vector field generating it.
- $F_2(x, y, z) = z^2 - 2\mu y$ and $F_3(x, y, z) := z^3 - 3\mu yz + 3\mu^2 z$ are invariants.

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- $F_2(x, y, z) = z^2 - 2\mu y$ and $F_3(x, y, z) := z^3 - 3\mu yz + 3\mu^2 z$ are invariants.
- For $F(x, y, z) := (F_3(x, y, z), F_2(x, y, z), z)$, $F \circ \Phi_t \circ F^{-1}$ is a translation by $(0, 0, \mu t)$.

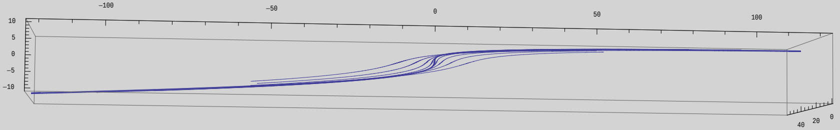

Figure: A number of orbits drawn horizontally.

Lorentzian analog of parabolic neighborhoods

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- Take a line l with direction $(a, 0, c)$, $a, c > 0$ in the timelike direction $2ac > 0$.
- $\Psi(t, s) = \Phi_t(l(s))$ for $l(s) = (0, y_0, 0) + s(a, 0, c)$ for $(0, y_0, 0) \in P_T$ where $T = -2\mu y_0$.

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- For $y_0 < \mu \frac{a}{c}$, $\Psi(t, s)$ is a proper imbedding to Φ_t invariant ruled surface.

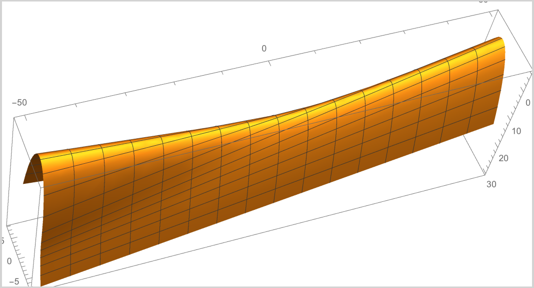

Figure: The ruled surface

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