1 Introduction

About this lecture

- Russell's theory of Description
- Predicate and names
- Quantifiers and variables
- Formation rules
- Models
- Refutation trees of predicate logic
- Identity
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer
- A mathematical introduction to logic, H. Enderton, Academic Press.
- Whitehead, Russell, Principia Mathematica (our library). (This could be a project idea.)
- `http://plato.stanford.edu/contents.html` has much resource. See "Descriptions".
- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Monadic Predicate Calculus".
- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.
- `http://logic.philosophy.ox.ac.uk/`. See "Predicate Calculus" in Tutorial.

Some helpful references

- `http://en.wikipedia.org/wiki/Truth_table`,
- `http://logik.phl.univie.ac.at/~chris/gateway/formular-uk-zentral.html`, complete (i.e. has all the steps)
- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

2 Russell's theory of Description

Russell's theory of Description

- Often we use sentences like "Tom is a man". "A person of African descent is the President of America."
- $M(x)$: x is a man, $B(x)$: x is of African descent. $P(x)$: x is the President of America.
- We have $M(Tom)$.
- There exists x s.t. $B(x) \to P(x)$ hold.
- How does one analyze such arguments logically.
- A statement such as a is a KAIST student.
- This is a description $K(a)$.

Russell's theory of Description

- Is the statement "The present king of Korea is of Japanese descent" correct?
- There exists x such that $K(x) \to J(x)$.
- There exists x such that $K(x) \wedge J(x)$.
- These two are logically different.
- Of course the theory of descriptions has some controversies as well.

3 Quantifiers and variables

Quantifiers

- Universal quantifier $\forall x$.
- Existential quantifier $\exists x$.
- There exists x such that if x is $K(x)$, then x is $J(x)$.
- $\exists x, K(x) \to J(x)$.
- Every body in KAIST has a course that he takes and which he hates.
- $\forall x(K(x) \to \exists c(T(x,c) \wedge H(x,c)))$.

Examples

- Nobody wish to get close to some one with H1N1 virus.
- $\forall x(H1(x) \rightarrow \neg(\exists y C(y,x))$.
- If any one in the dorm has a friend who has the measles, then everyone in the room will be quarantined.
- $(\exists x(D(x) \wedge (\exists y(F(y,x) \wedge M(y))))) \rightarrow (\forall z(D(z) \rightarrow Q(z)))$.

Quantifier negation laws

- $\neg\exists x P(x) \leftrightarrow \forall x \neg P(x)$.
- $\neg\forall x P(x) \leftrightarrow \exists x \neg P(x)$.
- This will be proved later. (See also HTP)
- Every body has a relative he does not like.
- Negate this statement.
- $\forall x(\exists y(R(x,y) \wedge \neg L(x,y)))$.
- $\neg\forall x(\exists y(R(x,y) \wedge \neg L(x,y)))$.
- $\exists x \neg(\exists y(R(x,y) \wedge \neg L(x,y)))$.
- $\exists x(\forall y \neg(R(x,y) \wedge \neg L(x,y)))$.
- $\exists x \forall y(\neg R(x,y) \vee L(x,y))$.
- $\exists x \forall y(R(x,y) \rightarrow L(x,y))$.
- There is someone who likes all his relatives.

Interchangible

- $\forall x \forall y$ interchangible to $\forall y \forall x$.
- $\exists x \exists y$ interchangible to $\exists y \exists x$.
- Other types are not interchangible.
- $\exists x \exists y(T(y,x) \wedge P(y,x))$.
- There is some one A who is a teacher of some one B and is younger than B.
- $\exists y \exists x(T(y,x) \wedge P(y,x))$
- There is some one B who is a student of some one A and is older than A.

Some other equivalences

- $\exists x(f \vee g) \leftrightarrow \exists x f \vee \exists x g$.
- $\forall x(f \wedge g) \leftrightarrow \forall x f \wedge \forall y g$.
- $\exists x(f \wedge g) \leftrightarrow (\exists x f) \wedge g$ if x does not occur as a free variable of g. And also $\exists x(f \vee g) \leftrightarrow (\exists x f) \vee g$
- $\forall x(f \vee g) \leftrightarrow (\forall x f) \vee g$ if x does not occur as a free variable of g. And also $\forall x(f \wedge g) \leftrightarrow (\forall x f) \wedge g$
- $\exists y f(x_1, ..., x_n, y) \leftrightarrow \exists z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
- $\forall y f(x_1, ..., x_n, y) \leftrightarrow \forall z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
- $\exists x f \leftrightarrow f$ if x is not a free variable of f.
- $\forall x f \leftrightarrow f$ if x is not a free variable of f.
- But $\exists x(E(x) \wedge T(x))$ is not equivalent to $(\exists x E(x)) \wedge (\exists x T(x))$.
- $\forall x(E(x) \vee T(x))$ is not equivalent to $(\forall x E(x)) \vee (\forall x T(x))$.

4 Predicate and name

Predicate and names

- Jones is a thief. $T(j)$.
- $T(x)$ x is a thief. j Jones.
- Bob loves Cathy.
- $L(b, c)$, $L(c, b)$.
- Cathy gave Fido to Bob.
- $G(c, f, b)$. $G(x, y, z)$. x gave y to z.

Predicate and names

- Jones likes everyone.
- $\forall x L(j, x)$.
- Jones likes a nurse.
- $\exists x(N(x) \wedge L(j, x))$.
- Jones likes every nurse.
- $\forall x(N(x) \rightarrow L(j, x))$.
- A nurse likes a mechanic.
- $\exists x \exists y((N(x) \wedge M(y)) \rightarrow L(x, y))$.

5 Formation rules

Formation rules

- Logical symbols:
  - Logical operators $\neg, \wedge, \vee, \rightarrow, \leftarrow$.
  - Quantifiers $\forall$, $\exists$.
  - Variables; letter $u, v, z, ...$
- Nonlogical symbols:
  - Names: $a, b, ..., t$.
  - Predicate: $A, B, C, ...$

Well formed formula

- Any atomic formula is a wff. P, $K(a)$, $J(a, b)$, so on.
- If ϕ is a wff, then so is $\neg\phi$.
- If ϕ and ψ are wffs, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, and $\phi \leftrightarrow \psi$.
- If ϕ is a wff containing a name letter α, then any formula of form $\forall\beta\phi^{\beta/\alpha}$ and $\exists\beta\phi^{\beta/\alpha}$ for a variable β are wff.
- Here, $\phi^{\beta/\alpha}$ means that we replace every or some occurance of α in ϕ with β.

Examples

- $F(a) \wedge G(a, b)$. a is fast and a is greater than b.
- $\forall x(F(x) \wedge G(x, b))$.
- $\exists y\forall x(F(x) \wedge G(x, y))$.
- There exists someone who is less than all the fast people.
- $\forall xL(x, z)$
- not wff.
- $\exists x\exists x(F(x) \wedge (\neg G(x)))$. This violates rules.

6 Models

Models

- Semantics or actual interpretations of symbols... i.e., universe A, B,... today's universe.... These could even be finitely many.
- These could form sets, but not necessarily so.
- Symbols: Model interpretations
- name letter: indiviual objects
- zero-place predicate letter: truth value T or F.
- one-place predicate letter: A class of objects.
- n-place predicate letter: a relation between n objects.
- Given a model M, it is possible that different simbols represent the same objects or relations.
- We try to avoid giving same letters to different objects or relations in models.

Truth value assignment

- A single letter. The truth value is the one directly supplied by the model.
- Predicate P. $P(a)$ is true if a belongs to the class of object denoted by P.
- $R(a, b, ..., g)$ is true if the relation hold between $a, b, .., g$ and is false if not.

Examples

- Universe: the class of all people.
- o Obama, h Hillary Clinton, c Bill Clinton, g George W. Bush: P the class of the 21st century U.S. Presidents. b people who own black dogs.
- $\forall x(Px \to Bx)$.
- $x = o$. T. $x = g$. T.
- $x = h$ or anyother person. T.
- Thus $\forall x(Px \to Bx)$ is true.
- Let P' be the class of 20th century president.
- Check $\forall x(P'x \to Bx)$.

α-variant of a model M

- M a model, and α a name letter (an external object)
- The α-variant of M is a model with the same universe as M and freely interpreting α as any object in M.
- A universal quantification $\forall\beta\phi$ is true in M if the wff $\phi^{\alpha/\beta}$ is true for every α-variant of M.
- An existential quantification $\exists\beta\phi$ is true in M if the wff $\phi^{\alpha/\beta}$ is true for some α-variant of M.
- If the wff $\phi^{\alpha/\beta}$ is true for no α-variant of M, then $\exists\beta\phi$ is false.
- Universe: all living creatures. B the class of blue things. W the class of winged horses.
- $\forall x(Wx \to Bx)$. Is this true?
- We can let α be any living creature. Then Wx is always false.

Examples

- Universe: the class of all positive integers
- E: the class of even integers, B relation bigger than
- $\forall x(Ex \to \forall yBxy)$.
- α-variant of M.,
- α odd. Then true.
- α even $\forall yBay$. False.
- Thus false.
- Example: $\forall y\exists xBxy$.
- True.

7 Refutation trees of predicate logic

Validity of predicate logic

- We would write some statements is valid if it is true for all models of the theory.
- We write $P, Q, \models R$ if $(P \wedge Q) \to R$ is true on every model of the theory.
- Example: $\exists x\forall yG(x,y) \models \forall y\exists xG(x,y)$ is valid.

- Example: $\forall y \exists x G(x, y) \models \exists x \forall y G(x, y)$ is invalid. (See 6.20, 6.21, 6.22)
- Note here the role of the models.
- In this book, we confuse $\models$ with $\vdash$.

Refutation trees of predicate logic

- One can use the refutation tree method for propositional logic for predicate logic also.
- This works by using negation rules for universal quantifiers and existential quantifiers. See Example 6.24.
- We will give rules for refutation trees for predicate logic.
- The rules can show the validity (i.e. the soundness of the rule.)
- However, rule may not detect invalidity (i.e. incompleteness of the rule). That is, sometimes, it won't give us counter-example.

Refutation trees of predicate logic example

- Prove $\forall x F(x) \rightarrow \forall x G(x), \neg \forall x G(x) \vdash \neg \forall x F(x)$.
- 1. $\forall x F(x) \rightarrow \forall x G(x)$. 2. $\neg \forall x G(x)$ 3. $\neg\neg \forall x F(x)$
- ✓ 1. $\forall x F(x) \rightarrow \forall x G(x)$. 2. $\neg \forall x G(x)$ 3. $\neg\neg \forall x F(x)$, 4(i) $\neg \forall x F(x)$ 4(ii) $\forall x G(x)$. $\rightarrow E$.1
- 2. $\neg \forall x G(x)$ 3. $\neg\neg \forall x F(x)$, 4(i) $\neg \forall x F(x)$ 5. (X) 4(ii) $\forall x G(x)$. $\rightarrow E$.1 5. (X).

Universal quantifier rule $\forall$.

- We have $\forall \beta \phi$ and a name letter α is on an open path containing it, write $\phi^{\alpha/\beta}$ at the bottom of that path.
- If no name letter appears on the open path, then choose some name letter α and write $\phi^{\alpha/\beta}$ at the bottom of that path.
- But do not check $\forall \beta \phi$. (Since we will use it many times.)

Example

- All university students are weak.
- Everyone is a university student.
- Alf is a university student.
- Thus, Alf is weak.

- $\forall x(Ux \to Wx), \forall x Ux \vdash Wa$.
- 1. $\forall x(Ux \to Wx)$, 2. $\forall x Ux$ 3. $\neg Wa$.
- 1. $\forall x(Ux \to Wx)$, 2. $\forall x Ux$ 3. $\neg Wa$. 4. $Ua \to Wa$ (1 $\forall$.)
- 1. $\forall x(Ux \to Wx)$, 2. $\forall x Ux$ 3. $\neg Wa$. 4. $Ua \to Wa$ (1 $\forall$.) 5. Ua (2 $\forall$)
- 1. $\forall x(Ux \to Wx)$, 2. $\forall x Ux$ 3. $\neg Wa$. 4. ✓ $Ua \to Wa$ (1 $\forall$.) 5. Ua (2 $\forall$) 6. (i) $\neg Ua$ (4. $\to$) 6. (ii) Wa (4 $\to$).
- 1. $\forall x(Ux \to Wx)$, 2. $\forall x Ux$ 3. $\neg Wa$. 5. Ua (2 $\forall$) 6. (i) $\neg Ua$ (4. $\to$) 7. (X) 6. (ii) Wa (4 $\to$). 7 (X)

More rules.

- Existential quantification $\exists$: $\exists \beta \phi$ check it and choose α not anywhere and write $\phi^{\alpha/\beta}$.
- Negated existential quantification $\neg\exists$: $\neg\exists\phi$ check it and write $\forall\neg\phi$.
- Negated universal quantification $\neg\forall$: $\neg\forall\phi$ check it and write $\exists\neg\phi$.
- These two are equivalences.

Example

- Holmes, if any one can trap Moriarty, he can. Holmes can't. No-one can.
- $\forall x Txm \to Thm, \neg Thm, \vdash \neg\exists x Txm$.
- 1. $\forall x Txm \to Thm$, 2. $\neg Thm$, 3. $\neg\neg\exists x Txm$.
- 1. $\forall x Txm \to Thm$, 2. $\neg Thm$, 3. ✓$\neg\neg\exists x Txm$. 4. $\exists x Txm$.
- 1. $\forall x Txm \to Thm$, 2. $\neg Thm$, 4. $\exists x Txm$. 5 $Tmm \to Thm$ (1 $\forall$).
- 1. $\forall x Txm \to Thm$, 2. $\neg Thm$, 4. $\exists x Txm$. 5 ✓$Tmm \to Thm$ (1 $\forall$). 6. (i) $\neg Tmm$ (5 $\to$) 6.(ii) Thm. (5 $\to$). (X 2, 6)
- 1. $\forall x Txm \to Thm$, 2. $\neg Thm$, 4. $\exists x Txm$. 5 $Tmm \to Thm$ (1 $\forall$). 6.(ii) Thm. (5 $\to$). (X 2, 6) 6. (i) $\neg Tmm$ (5 $\to$) 7. Tcm (4 $\exists$). 8. $Tcm \to Thm$ (1 $\forall$) 9 (i) $\neg Tcm$ (X, 4) (ii) Thm (X 2). (8 $\to$).

Example

- There is some one who loves someone. Then there exists someone who loves himself.
- $\exists x \exists y Lxy \vdash \exists x Lxx$.
- 1 $\exists x \exists y Lxy$. 2. $\neg \exists x Lxx$.
- 1 ✓$\exists x \exists y Lxy$. 2. $\neg \exists x Lxx$. 3. $\exists y Lay$ (1 ∃).
- 1 ✓$\exists x \exists y Lxy$. 2. $\neg \exists x Lxx$. 3. ✓$\exists y Lay$ (1 ∃). 4. Lab. (4 ∃.)
- 2. ✓$\neg \exists x Lxx$. 4. Lab. (4 ∃.) 5. $\forall x \neg Lxx$.
- 4. Lab. (4 ∃.) 5. $\forall x \neg Lxx$. 6. $\neg Laa$ (5 ∀).
- 4. Lab. 5. $\forall x \neg Lxx$. 6. $\neg Laa$ (5 ∀). 7. $\neg Lbb$ (5 ∀)...
- Invalid.

8 Identity

Identity

- We can introduce the identity symbols $=$ to predicate logic.
- $=$ indicates two objects are the "same".
- Symbols c Samuel Clemens, h Huckleberry Finn the Novel, t Mark Twain.
- Mark Twain is not Samuel Clemens. $\neg(t = c)$ or $t \neq c$.
- Only Mark Twain wrote Huckelberry Finn. $\forall x(Wxh \rightarrow x = t)$.
- Mark Twain is the best American writer $At \wedge (\forall x(Ax \wedge \neg x = t) \rightarrow Btx)$.

Refutation tree rules for Identity

- Identity ($=$) rule: $\alpha = \beta$ occurs. Then we can replace from ϕ any number of α with β and vice versa at the bottom of the path.
- Negated Identity Rule ($\neg =$): $\neg \alpha = \alpha$ occurs. Then we can close the path containing it.

Example

- $\vdash \forall x \forall y(x = y \to y = x)$.
- 1. $\neg\forall x \forall y(x = y \to y = x)$.
- 1. $\checkmark\neg\forall x \forall y(x = y \to y = x)$. 2. $\exists x \neg\forall y(x = y \to y = x)$.
- 1. $\checkmark\neg\forall x \forall y(x = y \to y = x)$. 2. $\checkmark\exists x \neg\forall y(x = y \to y = x)$. 3. $\neg\forall y(a = y \to y = a)$. (2 $\exists$.)
- 3. $\checkmark\neg\forall y(a = y \to y = a)$. (2 $\exists$.) 4. $\exists y \neg(a = y \to y = a)$. (3 $\neg\forall$).
- 4. $\checkmark\exists y \neg(a = y \to y = a)$. 5. $\neg(a = b \to b = a)$.
- 5. $\checkmark\neg(a = b \to b = a)$. 6. $a = b$ (5 $\neg \to$) 7. $\neg b = a$ (5 $\neg \to$).
- 6. $a = b$ (5 $\neg \to$) 7. $\neg b = a$ (5 $\neg \to$). 8. $\neg a = a$. $6, 7 =$. X.
- valid.

Some other equivalences (Repeated)

- $\exists x(f \vee g) \leftrightarrow \exists x f \vee \exists x g$.
- $\forall x(f \wedge g) \leftrightarrow \forall x f \wedge \forall y g$.
- $\exists x(f \wedge g) \leftrightarrow (\exists x f) \wedge g$ if x does not occur as a free variable of g. And also $\exists x(f \vee g) \leftrightarrow (\exists x f) \vee g$
- $\forall x(f \vee g) \leftrightarrow (\forall x f) \vee g$ if x does not occur as a free variable of g. And also $\forall x(f \wedge g) \leftrightarrow (\forall x f) \wedge g$
- $\exists y f(x_1, ..., x_n, y) \leftrightarrow \exists z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
- $\forall y f(x_1, ..., x_n, y) \leftrightarrow \forall z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
- $\exists x f \leftrightarrow f$ if x is not a free variable of f.
- $\forall x f \leftrightarrow f$ if x is not a free variable of f.
- But $\exists x(E(x) \wedge T(x))$ is not equivalent to $(\exists x E(x)) \wedge (\exists x T(x))$.
- $\forall x(E(x) \vee T(x))$ is not equivalent to $(\forall x E(x)) \vee (\forall x T(x))$.