Logic and the set theory

Lecture 11,12: Quantifiers (The set theory) in How to Prove It.

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Fall semester, 2012

About this lecture

- Sets (HTP Sections 1.3, 1.4)

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

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- $x \in B$. What does this mean?

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- For any collection of sets, there exists a unique set that contains all the elements that belong to at least one set in the collection. (Union)

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- Let $P(x, y)$ be a property that for every x, there exists unique y so that $P(x, y)$ holds. Then for every set A, there is a set B such that for every $x \in A$, there is $y \in B$ so that $P(x, y)$ holds. (Substitution)

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- Zermelo-Fraenkel theory has more axioms...The axiom of foundation, the axiom of choice.(ZFC)

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- $\emptyset$ is the empty set.

Operations on sets

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- $A - B = \{x | x \in A \wedge x \notin B\}$.
- $A = \emptyset$ if and only if $\neg \exists x(x \in A)$.

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- Thus, $x \in A \cup (B \cap C) \leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$.

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- Thus, $x \in A \cup (B \cap C) \leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$.
- One can use Venn diagrams....

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- Use logic to find examples.

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- Find the counter-example...(Using what?)

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- $\forall a(a \geq -2 \leftrightarrow \exists x \in \mathbb{R}(ax^2 + 4x - 2 = 0))$.

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- Use $\mathbb{R}$.
- $\forall a(a \geq -2 \leftrightarrow \exists x \in \mathbb{R}(ax^2 + 4x - 2 = 0))$.
- Is this true? How does one verify this...

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- There exists an element of A not in B.

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- Given a set A, the power set is defined: $P(A) = \{x | x \subset A\}$.
- $x \in P(A)$ is equivalent to $x \subset A$ and to $\forall y (y \in x \rightarrow y \in A)$.

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- $P(A) \subset P(B)$. Analysis

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The power set

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- If $A \subset B$, then is $P(A) \subset P(B)$?
- To check this what should we do? Use our inference rules....

- $A \subset B \vdash \forall x((\forall y(y \in x \rightarrow y \in A)) \rightarrow (\forall y(y \in x \rightarrow y \in B)))$.

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- 7. $(\forall y(y \in a \to y \in A)) \to \forall y(y \in a \to y \in B)$. 2-6

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- Alternate notations: $\mathcal{F} = \{A_i | i \in I\}$.
- $\bigcap \mathcal{F} = \bigcap_{i \in I} A_i = \{x | \forall i \in I (x \in A_i)\}$.

- $\mathcal{F} = \{C_s | s \in S\}$ a family of sets.
- Define $\bigcup \mathcal{F}$ as the set of elements in at least one element of $\mathcal{F}$.
- $\bigcup \mathcal{F} = \{x | \exists A(A \in \mathcal{F} \wedge x \in A)\} = \{x | \exists A \in \mathcal{F}(x \in A)\}$.
- Define $\bigcap \mathcal{F}$ as the set of common elements of elements of $\mathcal{F}$.
- $\bigcap \mathcal{F} = \{x | \forall A(A \in \mathcal{F} \rightarrow x \in A)\} = \{x | \forall A \in \mathcal{F}(x \in A)\}$.
- Alternate notations: $\mathcal{F} = \{A_i | i \in I\}$.
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- $\bigcup \mathcal{F} = \bigcup_{i \in I} A_i = \{x | \exists i \in I(x \in A_i)\}$.

Example

- $x \in P(\bigcup \mathcal{F})$. Analysis:

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- $\forall y(y \in x \to y \in \bigcup \mathcal{F})$.
- $\forall y(y \in x \to \exists A \in \mathcal{F}(y \in A))$.

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- Prove that $x \in \mathcal{F} \vdash x \in P(\bigcup \mathcal{F})$.

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- 3.: $\exists A \in \mathcal{F}(a \in A)$.
- 4. $a \in x \rightarrow (\exists A \in \mathcal{F}(a \in A))$ 2-3.

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- $x \subset \bigcup \mathcal{F}$.
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- 1. $x \in \mathcal{F}$. A.
- 2.: $a \in x$ H.
- 3.: $\exists A \in \mathcal{F}(a \in A)$.
- 4. $a \in x \to (\exists A \in \mathcal{F}(a \in A))$ 2-3.
- 5. $\forall y(y \in x \to (\exists A \in \mathcal{F}(y \in A)))$

Example

- $x \in P(\bigcup \mathcal{F}) \vdash x \in \mathcal{F}$. Is this valid?
- Try to use refutation tree test.
- $x \in P(\bigcup \mathcal{F})$. $x \notin \mathcal{F}$.

1 $\forall y(y \in x \to \exists A \in \mathcal{F}(y \in A))$.

2 $x \notin \mathcal{F}$. negation first.

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3 $a \in x \to \exists A \in \mathcal{F}(a \in A)$.

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2 $x \notin \mathcal{F}$.
3 check $a \in x \rightarrow \exists A \in \mathcal{F}(a \in A)$.
4 (i) $a \notin x$ 4(ii) $\exists A(a \in A \wedge A \in \mathcal{F})$.

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1 $\forall y(y \in x \rightarrow \exists A \in \mathcal{F}(y \in A))$.
2 $x \notin \mathcal{F}$.
4 (i) $a \notin x$ open 4(ii) check $\exists A(a \in A \wedge A \in \mathcal{F})$
5 (ii) $a \in A_0$
6 (ii) $A_0 \in \mathcal{F}$.

- How do one obtain a counter-example? $x \notin \mathcal{F}$ and $a \notin x$.
- $\mathcal{F} = \{\{1,2\},\{1,3\}\}$. $x = \{1,2,3\}$. $a = 4$.
- $\mathcal{F} = \{\{1,2\},\{1,3\}\}$. $x = \{1,2,3\}$. $a = 3$. $a \in \{1,3\}$. $\{1,3\} \in \mathcal{F}$.