Logic and the set theory

Lecture 13: Proofs in How to Prove It.

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

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- Because of these differences of standards, it is often very hard to communicate with other fields.

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  $\neg P, P \wedge Q.P \vee Q.P \rightarrow Q, P \leftrightarrow Q, \forall x P(x), \exists x P(x), \exists! x P(x)$.

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- for a goal of form:
  $\neg P, P \wedge Q, P \vee Q, P \rightarrow Q, P \leftrightarrow Q, \forall x P(x), \forall n \in \mathbb{N} P(n), \exists x P(x), \exists! x P(x)$.

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- The basic assumption we will have in mathematics is the ZFC.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ are the important sets.

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- First method: Assume P and prove Q. Or add P to the list of hypothesis and prove Q.

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Given	Goal
$- - --$	$P \to Q$
$- - --$	
- Change to

Given	Goal
$- - --$	Q
$- - --$	
P	

- Example $0 < a < b \to a^2 < b^2$.

- Example $0 < a < b \rightarrow a^2 < b^2$.
-

Given	Goal
$- - -$	$0 < a < b \rightarrow a^2 < b^2$
$- - - -$	

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Given	Goal
$---$	$0 < a < b \to a^2 < b^2$
$-----$	

- Change to

Given	Goal
$-----$	$a^2 < b^2$
$-----$	
$0 < a < b$	

- Example $0 < a < b \to a^2 < b^2$.
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Given	Goal
$---$	$0 < a < b \to a^2 < b^2$
$-----$	

- Change to

Given	Goal
$-----$	$a^2 < b^2$
$-----$	
$0 < a < b$	

-

Given	Goal
$0 < a < b$	$a^2 < b^2$
$0 < a^2 < ab$	
$0 < ab < b^2$	

To prove $P \to Q$

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Given	Goal
$- - --$	$P \to Q$
$- - --$	

- Change to

Given	Goal
$- - --$	$\neg P$
$- - --$	
$\neg Q$	

- Example: Let $a > b$. Then if $ac \leq bc$, then $c \leq 0$.

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Given	Goal
a, b, c are real numbers	$(ac \leq bc) \rightarrow (c \leq 0)$
$a > b$	

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a, b, c are real numbers	$(ac \leq bc) \rightarrow (c \leq 0)$
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Given	Goal
a, b, c are real numbers	$ac > bc$
$a > b$	
$c > 0$	

Write this in English

- Theorem: Let $a > b$. Then if $ac \leq bc$, then $c \leq 0$.

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- Theorem: Let $a > b$. Then if $ac \leq bc$, then $c \leq 0$.
- Proof: We will prove this by contrapositives. To prove $ac \leq bc \to c \leq 0$. It is sufficient to prove $c > 0 \to ac > bc$. Suppose $c > 0$. Then $ac > bc$ by $a > b$. □

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$A \cap C \subset B$	$a \notin A - B$
$a \in C$	

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- We change $a \notin A - B$.
- $a \notin A - B \leftrightarrow \neg(a \in A \wedge b \notin B). \leftrightarrow (a \notin A \vee a \in B). \leftrightarrow (a \in A \rightarrow a \in B)$.

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Given	Goal
$A \cap C \subset B$	$a \in A \rightarrow a \in B$
$a \in C$	

Given	Goal
$A \cap C \subset B$	$a \in B$
$a \in C$	
$a \in A$	

Given	Goal
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$a \in C$	
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Given	Goal
$A \cap C \subset B$	$a \in B$
$a \in C$	
$a \in A$	

- Theorem: Suppose that $A \cap C \subset B$ and $a \in C$. Prove $a \notin A - B$.
- Proof: To show $a \notin A - B$, it is equivalent to show $a \in A \to a \in B$. (See above). Assume $a \in A$. Since $A \cap C \subset B$ and $a \in C$, it follows that $a \in B$. □

To prove a goal of the form $\neg P$.

- Second method: Assume P and find a contradiction:

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- As above: Show $A \cap C \subset B$, $a \in C$. Prove $a \notin A - B$.

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Given	Goal
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$a \in C$	

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Given	Goal
$A \cap C \subset B$	$a \notin A - B$
$a \in C$	

-

Given	Goal
$A \cap C \subset B$	contradiction
$a \in C$	
$a \in A - B$	

To prove a goal of the form $\neg P$.

-

Given	Goal
$A \cap C \subset B$	contradiction
$a \in C$	
$a \in A - B$	
$a \in (A \cap C) - B$	
$a \in \emptyset$	

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Given	Goal
$\neg P$	*contradiction*
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Given	Goal
$\neg P$	*contradiction*
— — ——	

- Change to

Given	Goal
$\neg P$	P
— — ——	

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Given	Goal
$\neg P$	*contradiction*
— — ——	

- Change to

Given	Goal
$\neg P$	P
— — ——	

- Second method: re-express in some other form (positive form)

To use the given of the form $P \to Q$

- Use modus ponens $P, P \to Q \vdash Q$.

To use the given of the form $P \rightarrow Q$

- Use modus ponens $P, P \rightarrow Q \vdash Q$.
- Use modus tollens $P \rightarrow Q, \neg Q \vdash \neg P$.

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Given	Goal
$A \subset B$	$b \notin B$
$a \in A$	
$\neg(a \in B \wedge b \in B)$	

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$A \subset B$	$b \notin B$
$a \in A$	
$\neg(a \in B \wedge b \in B)$	

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Given	Goal
$A \subset B$	$b \notin B$
$a \in A$	
$(a \in B \to b \notin B)$	

Given	Goal
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$a \in A$	
$(a \in B \to b \notin B)$	
$a \in B$	

Given	Goal
$A \subset B$	$b \notin B$
$a \in A$	
$(a \in B \to b \notin B)$	
$a \in B$	

- Theorem: Suppose $A \subset B$, $a \in A$, and a and b are not both elements of B. Then $b \notin B$.

-

Given	Goal
$A \subset B$	$b \notin B$
$a \in A$	
$(a \in B \to b \notin B)$	
$a \in B$	

- Theorem: Suppose $A \subset B$, $a \in A$, and a and b are not both elements of B. Then $b \notin B$.
- Proof: Since a and b are not both elements of B, it follows that if a is an element of B, then b is not an element of B. Since $a \in A$, we have $a \in B$. Thus b is not an element of B. □

To show a goal of the form $\forall x P(x)$

- We introduce some arbitrary variable x in the assumption and prove $P(x)$.

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Given	Goal
– – – –	$\forall x P(x)$
– – – –	

-

Given	Goal
– – – –	$P(x)$
– – – –	

x is an arbitrary variable.

Examples

- A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$.

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Given	Goal
$A - B \subset C$	$A - C \subset B$

-

Given	Goal
$\forall x(x \in A - B \rightarrow x \in C)$	$\forall x(x \in A - C \rightarrow x \in B)$

Examples

- A, B, C are sets. $A - B \subset C$. Prove $A - C \subset B$.
- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ A - B \subset C & A - C \subset B \end{array}$$
- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & \forall x (x \in A - C \rightarrow x \in B) \end{array}$$
- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ \forall x (x \in A - B \rightarrow x \in C) & x \in A - C \rightarrow x \in B \\ x \text{ arbitrary} & \end{array}$$

Examples

-

Given	Goal
$\forall x(x \in A - B \to x \in C)$	$x \in B$
x arbitrary	
$x \in A - C$	

Examples

Given	Goal
$\forall x(x \in A - B \rightarrow x \in C)$	$x \in B$
x arbitrary	
$x \in A - C$	

Given	Goal
$\forall x(x \in A - B \rightarrow x \in C)$	contradiction
$x \in A$	
$x \notin C$	
$x \notin B$	

Given	Goal
$\forall x(x \in A - B \rightarrow x \in C)$	$x \in C$
$x \in A$	
$x \notin C$	
$x \notin B$	

- $$\begin{array}{cc} \text{Given} & \text{Goal} \\ \forall x(x \in A - B \rightarrow x \in C) & x \in C \\ x \in A & \\ x \notin C & \\ x \notin B & \end{array}$$
- Read the English proof also.

To prove a goal of form $\exists x P(x)$

- We guess x and show $P(x)$.

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Given	Goal
$- - --$	$\exists x P(x)$
$- - --$	

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Given	Goal
$- - --$	$\exists x P(x)$
$- - --$	

-

Given	Goal
$- - --$	$P(x)$
$- - --$	

x the value you decided

- $\exists x, |x^2 - 1| < 1/2$.

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-

Given	Goal
$x \in \mathbb{R}$	$\exists x, \|x^2 - 1\| < 1/2$

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Given	Goal
$x \in \mathbb{R}$	$\exists x, \|x^2 - 1\| < 1/2$

-

Given	Goal
$x \in \mathbb{R}$	$\exists x, \|x^2 - 1\| < 1/2$
$x = 1.1$	$(x^2 = 1.21, \|x^2 - 1\| = 0.21 < 1/2)$

To use a given of form $\exists x P(x)$ or $\forall x P(x)$

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Given	Goal
$\mathcal{F} \cap \mathcal{G} \neq \emptyset$	$\forall x(x \in \bigcap \mathcal{F} \to x \in \bigcup \mathcal{G})$

Given	Goal
$\mathcal{F} \cap \mathcal{G} \neq \emptyset$	$x \in \bigcup \mathcal{G}$
$x \in \bigcap \mathcal{F}$	

Given	Goal
$\mathcal{F} \cap \mathcal{G} \neq \emptyset$	$x \in \bigcup \mathcal{G}$
$x \in \bigcap \mathcal{F}$	

Given	Goal
$\exists A(A \in \mathcal{F} \cap \mathcal{G})$	$\exists A \in \mathcal{G}(x \in A)$
$\forall A \in \mathcal{F}(x \in A)$	

Given	Goal
$\mathcal{F} \cap \mathcal{G} \neq \emptyset$	$x \in \bigcup \mathcal{G}$
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Given	Goal
$\exists A(A \in \mathcal{F} \cap \mathcal{G})$	$\exists A \in \mathcal{G}(x \in A)$
$\forall A \in \mathcal{F}(x \in A)$	

Given	Goal
$A_0 \in \mathcal{F}$	$\exists A \in \mathcal{G}(x \in A)$
$A_0 \in \mathcal{G}$	
$\forall A \in \mathcal{F}(x \in A)$	
$x \in A_0$	

Given	Goal
$A_0 \in \mathcal{F}$	$\exists A \in \mathcal{G}(x \in A)$
$A_0 \in \mathcal{G}$	
$\forall A \in \mathcal{F}(x \in A)$	
$x \in A_0$	(Use $A = A_0$)

Given	Goal
$A_0 \in \mathcal{F}$	$\exists A \in \mathcal{G}(x \in A)$
$A_0 \in \mathcal{G}$	
$\forall A \in \mathcal{F}(x \in A)$	
$x \in A_0$	(Use $A = A_0$)

- Theorem: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. $\mathcal{F} \cap \mathcal{G} = \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.

Given	Goal
$A_0 \in \mathcal{F}$	$\exists A \in \mathcal{G}(x \in A)$
$A_0 \in \mathcal{G}$	
$\forall A \in \mathcal{F}(x \in A)$	
$x \in A_0$	(Use $A = A_0$)

- Theorem: Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. $\mathcal{F} \cap \mathcal{G} = \emptyset$. Then $\bigcap \mathcal{F} \subset \bigcup \mathcal{G}$.
- Proof: Suppose $x \in \bigcap \mathcal{F}$. Since $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Let A_0 be the common element. Then $A_0 \in \mathcal{F}$. Thus, $x \in A_0$ as $A_0 \in \mathcal{F}$. Since $A_0 \in \mathcal{G}$, then $x \in \bigcup \mathcal{G}$. □

Proofs involving conjunctions and biconditionals

- To prove a goal of the form $P \wedge Q$: Prove P and Q separately.

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Proofs involving conjunctions and biconditionals

- To prove a goal of the form $P \wedge Q$: Prove P and Q separately.
- To use $P \wedge Q$: Regard as P and Q.
- To prove a goal $P \leftrightarrow Q$: Prove $P \to Q$ and $Q \to P$.
- To use $P \leftrightarrow Q$: Treat as two givens $P \to Q$ and $Q \to P$.

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.

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- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
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Given	Goal
$\forall x \neg P(x)$	contradiction
$\exists x P(x)$	

Example

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- Prove $\rightarrow$: $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$
-

Given	Goal
$\forall x \neg P(x)$	contradiction
$\exists x P(x)$	

-

Given	Goal
$\forall x \neg P(x)$	contradiction
$P(x_0)$	

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove $\rightarrow$: $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$
Given	Goal
$\forall x \neg P(x)$	contradiction
$\exists x P(x)$	
Given	Goal
$\forall x \neg P(x)$	contradiction
$P(x_0)$	
Given	Goal
$\forall \neg P(x)$	contradiction
$P(x_0)$	
$\neg P(x_0)$	

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.

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- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove $\leftarrow$: $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
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-

Given	Goal
$\neg \exists x P(x)$	$\forall x \neg P(x)$

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove $\leftarrow$: $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$
-

Given	Goal
$\neg \exists x P(x)$	$\forall x \neg P(x)$

-

Given	Goal
$\neg \exists x P(x)$	$\neg P(x)$
x arbitrary	

Example

- Prove $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Prove $\leftarrow$: $\neg \exists x P(x) \rightarrow \forall x \neg P(x)$
-

Given	Goal
$\neg \exists x P(x)$	$\forall x \neg P(x)$

-

Given	Goal
$\neg \exists x P(x)$	$\neg P(x)$
x arbitrary	

-

Given	Goal
$\neg \exists x P(x)$	contradiction
x arbitrary	
$P(x)$	

Given	Goal
$\neg\exists x P(x)$	$\exists x P(x)$
x arbitrary	
$P(x)$	

Given	Goal
$\neg\exists x P(x)$	$\exists x P(x)$
x arbitrary	
$P(x)$	

- Theorem: $\forall x \neg P(x) \leftrightarrow \neg\exists x P(x)$.

Given	Goal
$\neg\exists x P(x)$	$\exists x P(x)$
x arbitrary	
$P(x)$	
- Theorem: $\forall x \neg P(x) \leftrightarrow \neg\exists x P(x)$.
- Proof: $(\rightarrow)$ Suppose $\forall x \neg P(x)$ and suppose $\exists x P(x)$. We choose x_0 such that $P(x_0)$ is true. Since $\forall x \neg P(x)$, we know $\neg P(x_0)$. This is a contradiction. Thus, $\forall x \neg P(x) \rightarrow \neg\exists x P(x)$.

Given	Goal
$\neg \exists x P(x)$	$\exists x P(x)$
x arbitrary	
$P(x)$	

- Theorem: $\forall x \neg P(x) \leftrightarrow \neg \exists x P(x)$.
- Proof: ($\rightarrow$) Suppose $\forall x \neg P(x)$ and suppose $\exists x P(x)$. We choose x_0 such that $P(x_0)$ is true. Since $\forall x \neg P(x)$, we know $\neg P(x_0)$. This is a contradiction. Thus, $\forall x \neg P(x) \rightarrow \neg \exists x P(x)$.
- Proof: ($\leftarrow$) Suppose $\neg \exists x P(x)$. Let x be arbitrary. Suppose that $P(x)$. Then $\exists x P(x)$. This is a contradiction. Thus $\neg P(x)$ is true. Since x was arbitrary, we have $\forall x \neg P(x)$. $\square$