Logic and the set theory

Lecture 15: Relations in How to Prove It.

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About this lecture

- Ordered pairs and Cartesian products

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- More about relations

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- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

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- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

Cartesian products

- A, B sets. $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.
- $\mathbb{R} \times \mathbb{R}$ Cartesian plane (Introduced by Descartes,.. used by Newton) First algebraic interpretation of curves...
- $P(x, y)$ The truth set of $P(x, y) = \{(a, b) \in A \times B | P(a, b)\}$.
- $x + y = 1$: $\{(a, b) \in \mathbb{R} \times \mathbb{R} | a + b = 1\}$.

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$A \times \emptyset = \emptyset \times A = \emptyset$.

Proof of 1

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Proof of 1

- $\subset$ part only

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$$\begin{array}{cc} \text{Given} & \text{Goal} \\ A, B, C & (a,b) \in (A \times B) \wedge (a,b) \in (A \times C) \\ (a,b) \in A \times (B \cap C) & \end{array}$$

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- If S is a relation from B to another set C, then $S \circ R := \{(a, c) \in A \times C | \exists b \in B((a, b) \in R \wedge (b, c) \in S)\}$.

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- $= \{(r, c) \in R \times C|$ The student s lives in a room r and enrolled in course c.$\}$.
- $= \{(r, c) \in R \times C|$ Some student living in a room r is enrolled in course c.$\}$.

Theorem

- $(R^{-1})^{-1} = R$.
- $Dom(R^{-1}) = Ran(R)$.
- $Ran(R^{-1}) = Dom(R)$.
- $T \circ (S \circ R) = (T \circ S) \circ R$.
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. *(note order)*

Proof of 5

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R, S	$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

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Proof of 5 continued

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2. R *is symmetric iff* $R^{-1} = R$.
3. R *is transitive iff* $R \circ R \subset R$.