Logic and the set theory

Lecture 17: Functions in How to Prove It.

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About this lecture

- Functions

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- One to one and onto

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- Inverse of functions

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- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://KLMS.kaist.ac.kr`

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- Images and inverse images
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://KLMS.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.

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- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

Functions

- Let F be a relation from A to B. F is said to be a *function* from A to B if

$$\forall a \in A \exists! b \in B((a, b) \in F).$$

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- If $(a, b) \in f$, we write $b = f(a)$ the value of a.
- Examples:

$$f = \{(s, p) | \text{ Prof p is the advisor of the student s } \}.$$

Functions

Theorem

$f, g : A \to B$. If $\forall a \in A(f(a) = g(a))$, then $f = g$.

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Proof.

Given $f(a) = g(a)$ for all $a \in A$, we show $f \subset g$ and $g \subset f$. Given $\forall a \in A, f(a) = g(a)$, we show $(a, b) \in f \to (a, b) \in g$ first. But $(a, b = f(a)) = (a, b = g(a))$. Thus clear. We show $(a, b) \in g \to (a, b) \in f$ similarly. □

Composition

- $Ran(f) = \{f(a)|a \in A\}$. $Dom(f) = A$.

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- Given $f : A \to B$ and $g : B \to C$, we define $g \circ f$ as relation.

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- Given $f : A \to B$ and $g : B \to C$, we define $g \circ f$ as relation.

Theorem

Let $f : A \to B, g : B \to C$. Then $g \circ f : A \to C$ and for all $a \in A$, $g \circ f(a) = g(f(a))$.

Proof

Given	Goal
$f : A \to B, g : B \to C$	$g \circ f : A \to C$
	$\forall a \in A \exists! c \in C((a, c) \in g \circ f)$

Proof

- Given: $f : A \to B, g : B \to C$

  Goal: $g \circ f : A \to C$
  $\forall a \in A \exists ! c \in C((a, c) \in g \circ f)$

- Given: $f : A \to B, g : B \to C$
  $a \in A$

  Goal: $\exists c \in C((a, c) \in g \circ f)$
  $\forall c_1 \in C \forall c_2 \in C$
  $(((a, c_1) \in g \circ f \wedge (a, c_2) \in g \circ f) \to c_1 = c_2)$

Proof

Given	Goal
$f : A \to B, g : B \to C$	$g \circ f : A \to C$
	$\forall a \in A \exists! c \in C((a, c) \in g \circ f)$

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$f : A \to B, g : B \to C$	$\exists c \in C((a, c) \in g \circ f)$
$a \in A$	$\forall c_1 \in C \forall c_2 \in C$
	$(((a, c_1) \in g \circ f \wedge (a, c_2) \in g \circ f) \to c_1 = c_2)$

- Existence part. clear.

Proof

- Uniqueness part:

Proof

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-

$$\begin{array}{cc} \text{Given} & \text{Goal} \\ f : A \to B, g : B \to C & c_1 = c_2 \\ a \in A & \\ c_1, c_2 \in C & \\ (a, c_1) \in g \circ f, (a, c_2) \in g \circ f & \end{array}$$

Proof

- Uniqueness part:
-

Given	Goal
$f : A \to B, g : B \to C$	$c_1 = c_2$
$a \in A$	
$c_1, c_2 \in C$	
$(a, c_1) \in g \circ f, (a, c_2) \in g \circ f$	

- $(a, b_1) \in f, (b_1, c_1) \in g, (a, b_2) \in f, (b_2, c_2) \in g$. Here $b_1 = b_2$ and hence $c_1 = c_2$.

One to one and onto

Definition

- $f : A \to B$ is said to be *one-to-one* if

$$\neg \exists a_1 \in A \exists a_2 \in A (f(a_1) = f(a_2) \wedge a_1 \neq a_2).$$

- f is *onto* if $\forall b \in B \exists a \in A (f(a) = b)$.

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Theorem

- *f is one-to-one iff $\forall a_1 \in A \forall a_2 \in A (f(a_1) = f(a_2) \to a_1 = a_2)$.*
- *f is onto iff $Ran(f) = B$.*

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- *f is one-to-one iff $\forall a_1 \in A \forall a_2 \in A (f(a_1) = f(a_2) \to a_1 = a_2)$.*
- *f is onto iff $Ran(f) = B$.*

Theorem

Let $f : A \to B, g : B \to C, g \circ f : A \to C$.

- *If f and g are both one-to-one, then so is $g \circ f$.*
- *If f and g are both onto, then so is $g \circ f$.*

Proof

- Injectivity only:

Given	Goal
$\forall a_1 \in B \forall a_2 \in B$	$\forall a_1 \in B \forall a_2 \in B$
$((f(a_1) = f(a_2) \to a_1 = a_2)$	$((g \circ f(a_1) = g \circ f(a_2) \to a_1 = a_2)$
$\forall b_1 \in B \forall b_2 \in B$	
$((g(b_1) = g(b_2) \to b_1 = b_2)$	

Proof

- Injectivity only:

Given	Goal
$\forall a_1 \in B \forall a_2 \in B$ $((f(a_1) = f(a_2) \to a_1 = a_2)$ $\forall b_1 \in B \forall b_2 \in B$ $((g(b_1) = g(b_2) \to b_1 = b_2)$	$\forall a_1 \in B \forall a_2 \in B$ $((g \circ f(a_1) = g \circ f(a_2) \to a_1 = a_2)$

- Injectivity only:

Given	Goal
$\forall a_1 \in B \forall a_2 \in B((f(a_1) = f(a_2) \to a_1 = a_2)$ $\forall b_1 \in B \forall b_2 \in B((g(b_1) = g(b_2) \to b_1 = b_2)$ $a_1 \in B, a_2 \in B, g \circ f(a_1) = g \circ f(a_2)$	$a_1 = a_2$

Proof

- Injectivity only:

Given	Goal
$\forall a_1 \in B \forall a_2 \in B$	$\forall a_1 \in B \forall a_2 \in B$
$((f(a_1) = f(a_2) \to a_1 = a_2)$	$((g \circ f(a_1) = g \circ f(a_2) \to a_1 = a_2)$
$\forall b_1 \in B \forall b_2 \in B$	
$((g(b_1) = g(b_2) \to b_1 = b_2)$	

- Injectivity only:

Given	Goal
$\forall a_1 \in B \forall a_2 \in B((f(a_1) = f(a_2) \to a_1 = a_2)$	$a_1 = a_2$
$\forall b_1 \in B \forall b_2 \in B((g(b_1) = g(b_2) \to b_1 = b_2)$	
$a_1 \in B, a_2 \in B, g \circ f(a_1) = g \circ f(a_2)$	

- $g(b_1) = g(b_2)$ for $b_1 = f(a_1), b_2 = f(a_2)$. Thus, $b_1 = b_2$ and hence $a_1 = a_2$.

Inverse of functions

Definition

The function that is one-to-one and onto are called *bijections* or *one-to-one correspondence*.

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The function that is one-to-one and onto are called *bijections* or *one-to-one correspondence*.

Theorem (5.3.1)

$f : A \to B$. If f is one-to-one and onto, then $f^{-1} : B \to A$ is also a one-to-one and onto function.

Inverse of functions

Proof.

- We show first that f^{-1} is a function. That is $\forall b \exists ! a \in A((b, a) \in f^{-1})$. This is divided into $\forall b \in B \exists a \in A((b, a) \in f^{-1})$ and

$$\forall b \in B \forall a_1 \in A \forall a_2 \in A(((b, a_1) \in f^{-1} \wedge (b, a_2) \in f^{-1}) \rightarrow a_1 = a_2).$$

□

Inverse of functions

Proof.

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- The first goal is equivalent to $\forall b \exists a \in A(b = f(a))$. Thus, f being onto implies this.

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Inverse of functions

Proof.

- We show first that f^{-1} is a function. That is $\forall b \exists ! a \in A((b, a) \in f^{-1})$. This is divided into $\forall b \in B \exists a \in A((b, a) \in f^{-1})$ and

$$\forall b \in B \forall a_1 \in A \forall a_2 \in A(((b, a_1) \in f^{-1} \wedge (b, a_2) \in f^{-1}) \rightarrow a_1 = a_2).$$

- The first goal is equivalent to $\forall b \exists a \in A(b = f(a))$. Thus, f being onto implies this.
- The second goal is equivalent to

$$\forall b \in B \forall a_1 \in A \forall a_2 \in A((b = f(a_1) \wedge b = f(a_2)) \rightarrow a_1 = a_2).$$

Hence, f being one-to-one implies this.

□

Theorem (5.3.2)

$f : A \to B$ and $f^{-1} : B \to A$ are functions. Then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

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Theorem (5.3.3)

$f : A \to B$.

- *If there is a function $g : B \to A$ such that $g \circ f = i_A$, then f is one-to-one.*
- *If there is a function $g : B \to A$ such that $f \circ g = i_B$, then f is onto.*

Proof.

- 1) We are given $f : A \to B, g : B \to A$ with $g \circ f = i_A$. Our goal is $\forall a_1 \in A \forall a_2 \in A((f(a_1) = f(a_2) \to a_1 = a_2)$. Suppose $a_1, a_2 \in A$ be arbitrary and $f(a_1) = f(a_2)$. Then $a_1 = i_A(a_1) = g \circ f(a_1) = g \circ f(a_2) = i_A(a_2) = a_2$.

□

Proof.

- 1) We are given $f : A \to B, g : B \to A$ with $g \circ f = i_A$. Our goal is $\forall a_1 \in A \forall a_2 \in A((f(a_1) = f(a_2) \to a_1 = a_2)$. Suppose $a_1, a_2 \in A$ be arbitrary and $f(a_1) = f(a_2)$. Then $a_1 = i_A(a_1) = g \circ f(a_1) = g \circ f(a_2) = i_A(a_2) = a_2$.
- 2) We are given $f \circ g = i_B$. We show $\forall b \exists a(b = f(a))$. Let $b \in B$ be arbitrary. Then we try to guess a. We have $b = f \circ g(b) = f(g(b))$. Let $a = g(b)$. Then $b = f(a) = f(g(b)) = b$.

□

Theorem (5.3.4)

The following statements are equivalent:

- *f is one-to-one and onto*
- *$f^{-1} : B \to A$ is a function.*
- *There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

Proof.

□

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- *There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

Proof.

- $1 \to 2$: Theorem 5.3.1.

□

Theorem (5.3.4)

The following statements are equivalent:

- *f is one-to-one and onto*
- *$f^{-1} : B \to A$ is a function.*
- *There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

Proof.

- $1 \to 2$: Theorem 5.3.1.
- $2 \to 3$ Theorem 5.3.2.

□

Theorem (5.3.4)

The following statements are equivalent:

- *f is one-to-one and onto*
- *$f^{-1} : B \to A$ is a function.*
- *There is a function $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.*

Proof.

- $1 \to 2$: Theorem 5.3.1.
- $2 \to 3$ Theorem 5.3.2.
- $3 \to 1$ Theorem 5.3.3.

□

Theorem (5.3.5)

Suppose that we have $f : A \to B$ and $g : B \to A$ and $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

Proof.

$\square$

Theorem (5.3.5)

Suppose that we have $f : A \to B$ and $g : B \to A$ and $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

Proof.

- By Theorem 5.3.4 ($2 \leftrightarrow 3$), $f^{-1} : B \to A$ is a function.

□

Theorem (5.3.5)

Suppose that we have $f : A \to B$ *and* $g : B \to A$ *and* $g \circ f = i_A$ *and* $f \circ g = i_B$. *Then* $g = f^{-1}$.

Proof.

- By Theorem 5.3.4 ($2 \leftrightarrow 3$), $f^{-1} : B \to A$ is a function.
- By Theorem 5.3.2, $f^{-1} \circ f = i_A$.

□

Theorem (5.3.5)

Suppose that we have $f : A \to B$ and $g : B \to A$ and $g \circ f = i_A$ and $f \circ g = i_B$. Then $g = f^{-1}$.

Proof.

- By Theorem 5.3.4 ($2 \leftrightarrow 3$), $f^{-1} : B \to A$ is a function.
- By Theorem 5.3.2, $f^{-1} \circ f = i_A$.
- Then $g = i_A \circ g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ i_B = f^{-1}$.

□

Images and Inverse images

- Given $f : A \to B$. $X \subset A$.

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- The image of X:

$$f(X) = \{f(x)|x \in X\} = \{b \in B|\exists x \in X(f(x) = b)\}.$$

Images and Inverse images

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- Inverse image of Y for $Y \subset B$:

$$f^{-1}(Y) = \{a \in A | f(a) \in Y\}.$$

Images and Inverse images

- Given $f : A \to B$. $X \subset A$.
- The image of X:

$$f(X) = \{f(x)|x \in X\} = \{b \in B|\exists x \in X(f(x) = b)\}.$$

- Inverse image of Y for $Y \subset B$:

$$f^{-1}(Y) = \{a \in A|f(a) \in Y\}.$$

Theorem (5.4.2)

$f : A \to B$, $W, X \subset A$. Then (1) $f(W \cap X) \subset f(W) \cap f(X)$. And (2) $f(W \cap X) = f(W) \cap f(X)$ if f is one to one.

Proof

- (1)

Given	Goal
$f : A \to B, W, X \subset A$	$f(W \cap X) \subset f(W) \cap f(X)$

Proof

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Given	Goal
$f : A \to B, W, X \subset A$	$f(W \cap X) \subset f(W) \cap f(X)$

-

Given	Goal
$f : A \to B, W, X \subset A$	$b \in f(W) \cap f(X)$
$b \in f(W \cap X)$	
$b = f(a), a \in W \cap X$	

Proof

- (2)

Given

$$f : A \to B, W, X \subset A$$

$$\forall a_1 \in A, \forall a_2 \in A$$
$$((f(a_1) = f(a_2)) \to a_1 = a_2$$

Goal

$$\forall b \in B \quad b \in f(W \cap X) \leftrightarrow$$
$$b \in f(W) \cap f(X)$$

Proof

- (2)

Given	Goal
$f : A \to B, W, X \subset A$	$\forall b \in B \quad b \in f(W \cap X) \leftrightarrow$
$\forall a_1 \in A, \forall a_2 \in A$	$b \in f(W) \cap f(X)$
$((f(a_1) = f(a_2)) \to a_1 = a_2$	

- $\to$ part is done in (1)

Proof

- (2)

Given

$$f : A \to B, W, X \subset A$$

$$\forall a_1 \in A, \forall a_2 \in A$$
$$((f(a_1) = f(a_2)) \to a_1 = a_2$$

Goal

$$\forall b \in B \quad b \in f(W \cap X) \leftrightarrow$$
$$b \in f(W) \cap f(X)$$

- $\to$ part is done in (1)
- (2)(ii)

Given

$$f : A \to B, W, X \subset A$$
$$(\forall a_1 \in A, \forall a_2 \in A$$
$$((f(a_1) = f(a_2)) \to a_1 = a_2)$$
$$b \in f(W) \cap f(X),$$
$$b = f(w) = f(x), w \in W, x \in X, w = x$$

Goal

$$b \in f(W \cap X)$$

Project homework

- See page 258,259 1-6. Do this individually (see klms for details)

Project homework

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- Due date is: November 30th (Friday)