Logic and the set theory

Lecture 19: The set theory

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About this lecture

- Axioms of the set theory

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  - Axiom of extension

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- Grading and so on in the moodle. Ask questions in moodle.

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- Sets for mathematics, F.W. Lawvere, R. Rosebrugh, Cambridge

Naive set theory (Zermelo-Fraenkel, ZFC)

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- The main reason that they exist is to aid in the proof and to follow the classical logic, and finally to avoid possible self-contradictions such as Russell's.
- The set theory can be characterized within the category theory.

Axiom of extension

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- Transitivity: $A \subset B$, $B \subset C$ Then $A \subset C$.
- Antisymmetry: $A \subset B$ and $B \subset A$. Then $A = B$.

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- A the set of all men. $\{x \in A | x \text{ is married }.\}$.

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- Example: $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, ...$

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- $\bigcup\{A_1, A_2, .., A_n\} = A_1 \cup A_2 \cup \cdots \cup A_n$.

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- $(A \cup B) \cup C = A \cup (B \cup C)$.
- $A \cup A = A$.
- $A \subset B \leftrightarrow A \cup B = B$.

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- Let $\emptyset$ be a family of subsets of E. Then $\bigcap_{X \in \emptyset} X = E$. (For no X, $x \in X$ is false.)
- Not specifying E gets you into trouble.

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- $A \cup \bigcap \mathcal{C} = \bigcap \mathcal{C}_2$ where $\mathcal{C}_2 = \{A \cup B | B \in \mathcal{C}\}$

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- $A \cap A' = \emptyset$ and $A \cup A' = E$.

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- $A \triangle B = (A - B) \cup (B - A)$. ($A + B$ in NS)

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- $P(X) = \{x | x \subset X\}$.
- $P(\emptyset) = \{\emptyset\}$.
- $P(\{a, b\}) = ?$

The power set

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- $\bigcup_{X \in \mathcal{C}} P(X) \subset P(\bigcup_{X \in \mathcal{C}} X)$.
- $\bigcap_{X \in P(E)} X = \emptyset$.
- $E \subset F \to P(E) \subset P(F)$.
- $E = \bigcup P(E)$.

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- $\{\emptyset, \{\emptyset, \{\emptyset\}\}\}$, ...
- Can we get an infinite set now?

Axiom of infinity

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- $0 := \emptyset$, $1 = 0^+ = \{0\} = \{\emptyset\}$, $2 = 1^+ = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$,

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- A *natural number* is an element of ω: $0, 1, 2, \ldots$.

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- (2) If A, B are both nonempty sets, then it is not possible that both $A \in B$ and $B \in A$ are true.

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- A is not a set but is a "class". (Von Neumann)

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- Axiom of Choice: The Cartesian product of a non-empty family of nonempty sets is nonempty.