Logic and the set theory

Lecture 21: The set theory: Review Sections 12-25

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About this lecture

- The axiom of choice

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- `http://us.metamath.org/mpegif/weth.mid` The music of proofs.

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- Application: Let X be a nonempty set. Then there exists a function $f : P(X) - \{\emptyset\} \to X$ so that $f(A) \in A$.

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  - If n and m are in ω, and if $n^+ = m^+$, then $n = m$.

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- If $n \notin E$, $f|n : n \to E - \{f(n)\}$ is one-to-one and onto. $E \subset n$ as $n \notin E$. $E - \{f(n)\}$ proper subset of n. Contradition.

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- If $n \in E$, n is equivalent to $E - \{n\}$. $n = E - \{n\}$ by induction hypothesis. Thus $E = n^+$. Contradiction.

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Theorem (Recursion)

Let X be a set, $a \in X$, and $f : X \to X$ be a function. Then there exists a function $u : \omega \to X$ such that $u(0) = a$ and $u(n^+) = f(u(n))$ for all $n \in \omega$.

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- Define $g : \mathcal{C} \to \mathcal{C}$ by $g(A) = A \cup \{f(X - A)\}$.

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- By Recursion theorem, there exists a function $U : \omega \to \mathcal{C}$ such that $U(0) = \emptyset$ and $U(n^+) = U(n) \cup \{f(X - U(n))\} = g(U(n))$. –(*)

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- (2) $v(n) \in U(n^+)$ for all $n \in \omega$ by (*) and (**).

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- $n^+ \leq m$. $U(n^+) \subset U(m)$. Thus $v(n) \subset U(n^+) \subset U(m)$ by (2) and (3).
- $v(m) \notin U(m)$. Thus $v(n) \neq v(m)$.

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Theorem (Zorn's lemma)

If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.

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- If $\mathcal{C}$ is a chain in χ, then $\bigcup \mathcal{C} \in \chi$: proof: omit.
- Suppose that we find a maximal element $\mathcal{F}$ in χ. Then $\mathcal{F}$ has an upper bound f_0. Then f_0 is a maximal element of X.

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- Let f be a choice function for χ: $f : P(X) - \{\emptyset\} \to X$ such that $f(A) \in A$ for all $A \in P(X)$.

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- Define $g : \chi \to \chi$ by if $\hat{A} - A \neq \emptyset$, then $g(A) = A \cup \{f(\hat{A} - A)\}$ and if $\hat{A} - A = \emptyset$, then $g(A) = A$.

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- We show that there exists $A \in \chi$ such that $g(A) = A$.

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- We show that there exists $A \in \chi$ such that $g(A) = A$.
- Then A is the element $\mathcal{F}$ we need.

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- We show that $\mathcal{T}_0$ is a chain : in the next frame.

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- We show that $\mathcal{T}_0$ is a chain : in the next frame.
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Proof continued

- A *tower* $\mathcal{T} \subset \chi$ is a subcollection such that
  - $\emptyset \in \mathcal{T}$.
  - If $A \in \mathcal{T}$, then $g(A) \in \mathcal{T}$.
  - If $\mathcal{C}$ is a chain in $\mathcal{T}$, then $\bigcup \mathcal{C} \in \mathcal{T}$.
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- Note: $g(A) \subset A$ and $g(A) - A$ cannot be more than a singleton.

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- The comparable sets in $\mathcal{T}_0$ constitutes a tower, and hence all sets in $\mathcal{T}_0$ are comparable. Thus, $\mathcal{T}_0$ is a chain.

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  - The existence of the choice functions implies the Axiom of Choice.

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- The axiom of choice $\rightarrow$ Zorn's lemma $\rightarrow$ The well-ordering theorem $\rightarrow$ The axiom of choice.

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Theorem (Burali-Forti paradox)

There is no set containing all ordinals.

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- A finite number is a cardinal as well as ω.

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- But the question still remains open in “some sense”, as a subject of “philosophy”.