Logic and the set theory

Lecture 6: Propositional Calculus: part 2

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About this lecture

- Notions of Inference

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- Hypothetical Rules

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- The soundness and the completeness of deductions.

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- We go over the last three Hypothetical Rules in Lecture 6.

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- Grading and so on in the moodle. Ask questions in moodle.

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- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Derivations in Sentential Calculus". (or SC Derivations.) and “The completeness of the SC rules.”

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- `http://jvrosset.free.fr/Goedel-Proof-Truth.pdf` “Does Godels incompleteness prove that truth transcends proof?”

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- Suppose that one proved a logical formula, which are not in the ten elementary rules. Then we can substitute the symbols with wffs and still obtain valid logical formula.

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- Suppose that one proved a logical formula, which are not in the ten elementary rules. Then we can substitute the symbols with wffs and still obtain valid logical formula.
- Example: $P \to Q, \neg Q \vdash \neg P$. (Modus Tollens (MT)).
- Substitution instance: P to $(R \vee S)$ and Q to $\neg C$. Then obtain $(R \vee S) \to \neg C, \neg\neg C. \vdash \neg(R \vee S)$.

Examples

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- Repeat or Reiteration (RE): $P \vdash Q \to P$.
- Contradiction (CON): $P, \neg P \vdash Q$.
- Disjunctive syllogism (DS): $P \vee Q, \neg P \vdash Q$.

Examples

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- 3. $Q \to S$. A.

Examples

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- 1. $P \vee Q$ A
- 2. $P \rightarrow R$ A.
- 3. $Q \rightarrow S$. A.
- 4.: P for $\rightarrow I$.

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- 2. $P \to R$ A.
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- 4.: P for $\to I$.
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- 6.: $R \vee S$ $\vee I$.

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- 4.: Q. 2.3. (CON)

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- $P \rightarrow (Q \rightarrow P)$.
- $(P \rightarrow (Q \rightarrow R)) \rightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$.
- $(((\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P))$.

Example

- Deduce (Prove) $\vdash ((\neg P \to \neg Q) \to (Q \to P))$.

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- Deduce (Prove) $\vdash ((\neg P \rightarrow \neg Q) \rightarrow (Q \rightarrow P))$.
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- 3. ::: $\neg P$

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- 1. :$\neg P \to \neg Q$ H. for $\to I$.
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- 4. ::: $\neg Q$. 1.3.

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- 7. : $Q \to P$. 2-5

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- 6. :: P
- 7. : $Q \to P$. 2-5
- 8. $\neg P \to \neg Q \to (Q \to P)$.

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- 2. :: $(P \to Q)$ for $\to I$.

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- 1. : $(P \to (Q \to R))$ for $\to I$.
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- 3. ::: P for $\to I$.
- 4. ::: $(Q \to R)$ 1.3.

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- 1. : $(P \to (Q \to R))$ for $\to I$.
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- 3. ::: P for $\to I$.
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- 5. ::: $P \to R$. 2.4.

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- 5. ::: $P \to R$. 2.4.
- 6. ::: R.

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- Deduce $\vdash (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$.
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- 6. ::: R.
- 7. :: $P \to R$. 3-6

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- 6. ::: R.
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- 8. : $(P \to Q) \to (P \to R)$. 2-7

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- 5. ::: $P \to R$. 2.4.
- 6. ::: R.
- 7. :: $P \to R$. 3-6
- 8. : $(P \to Q) \to (P \to R)$. 2-7
- 9. $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$ 1-8.

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- $\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$. DeMorgan's law. (DM)
- $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$. (DM)
- $P \vee Q \leftrightarrow Q \vee P$. Commutation (COM)

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- The equivalences can be verified by the truth table method or by deduction.

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- 8. $P \to Q$. 2-7.

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- See Theorem 4.3 of Cameron.
- If we go to a higher-order theory, this fails. (Gödel's incompleteness theorems: Theorem 5.8 in Cameron)