Logic and the set theory

Lecture 7, 8: Predicate Logic

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Fall semester, 2011

About this lecture

- Russell's theory of Description

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- Predicate and names

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- Quantifiers and variables

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- Grading and so on in the moodle. Ask questions in moodle.

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- `http://ocw.mit.edu/OcwWeb/Linguistics-and-Philosophy/24-241Fall-2005/CourseHome/` See "Monadic Predicate Calculus".

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- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.

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- `http://philosophy.hku.hk/think/pl/`. See Module: Predicate Logic.
- `http://logic.philosophy.ox.ac.uk/`. See "Predicate Calculus" in Tutorial.

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- `http://svn.oriontransfer.org/TruthTable/index.rhtml`, has xor, complete.

Formalizations

- All S are P: $\forall x, (Sx \to Px)$. (Not $\forall x, Sx \wedge Px$)

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- See Problems 6.1 and 6.2. page 132-133 Nolt.

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- This is a description $K(a)$.

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- Of course the theory of descriptions has some controversies as well. (If one accepts the theory, there are many implications.)

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- $\forall x(K(x) \to \exists c(T(x, c) \wedge H(x, c)))$.

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- $\forall x(H1(x) \to \neg(\exists y C(y,x))$.
- If any one in the dorm has a friend who has the measles, then everyone in the room will be quarantined.
- $(\exists x(D(x) \wedge (\exists y(F(y,x) \wedge M(y))))) \to (\forall z(D(z) \to Q(z)))$.

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- $\exists x \forall y(\neg R(x,y) \vee L(x,y))$.
- $\exists x \forall y(R(x,y) \rightarrow L(x,y))$.
- There is someone who likes all his relatives.

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- $\exists y \exists x (T(y, x) \wedge P(y, x))$
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- $\exists yf(x_1, ..., x_n, y) \leftrightarrow \exists zf(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.

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- Here, $\phi^{\beta/\alpha}$ means that we replace every or some occurance of α in ϕ with β.

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- We can let α be any living creature. Then *Wx* is always false.

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- In this book, we confuse $\models$ with $\vdash$.

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- We will give rules for refutation trees for predicate logic.
- The rules can show the validity (i.e. the soundness of the rule.)
- However, rule may not detect invalidity (i.e. incompleteness of the rule). That is, sometimes, it won't give us counter-example.

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- But do not check $\forall \beta \phi$. (Since we will use it many times.)

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1. $\forall x(Ux \to Wx)$,
2. $\forall x Ux$
3. $\neg Wa$.
5. Ua (2 $\forall$)
6. (i) $\neg Ua$ (4. $\to$) (X)
   (ii) Wa (4 $\to$). (X)

- valid

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- These two are equivalences.

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- Holmes, if any one can trap Moriarty, he can. Holmes can't. No-one can.
- $\forall x Txm \to Thm, \neg Thm, \vdash \neg\exists x Txm$.

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1 $\forall x Txm \rightarrow Thm,$

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5 $Tmm \rightarrow Thm$ (1 $\forall$).

1 $\forall x Txm \to Thm$,

2 $\neg Thm$,

4 $\exists x Txm$.

5 ✓ $Tmm \to Thm$ (1 $\forall$).

6 (i) $\neg Tmm$ (5 $\to$) 6.(ii) Thm. (5 $\to$). (X 2, 6)

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7 Tcm (4 $\exists$). 6(ii) Thm. (5 $\to$). (X 2, 6)

8 $Tcm \to Thm$ (1 $\forall$)

9 (i) $\neg Tcm$ (X, 4) (ii) Thm (X, 2). (8 $\to$).

- valid

Example

- There is some one who loves someone. Then there exists someone who loves himself.
- $\exists x \exists y Lxy \vdash \exists x Lxx$.

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4 Lab. (4 $\exists$.)

2 ✓$\neg\exists x Lxx$.

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5 $\forall x \neg Lxx$.

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4 Lab. (4 $\exists$.)

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6 . $\neg Laa$ (5 $\forall$).

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4 Lab.
5 $\forall x \neg Lxx$.
6 $\neg Laa$ (5 $\forall$).
7 $\neg Lbb$ (5 $\forall$)...
8 Invalid. (cannot do any more...)

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- Only Mark Twain wrote Huckelberry Finn. $\forall x(Wxh \to x = t)$.
- Mark Twain is the best American writer $At \wedge (\forall x(Ax \wedge \neg x = t) \to Btx)$.

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- Negated Identity Rule $(\neg =)$: $\neg \alpha = \alpha$ occurs. Then we can close the path containing it.

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We show $\vdash \forall x \forall y (x = y \to y = x)$.

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2 $\exists x \neg\forall y(x = y \rightarrow y = x)$.

Example

We show $\vdash \forall x \forall y (x = y \rightarrow y = x)$.

1. $\neg \forall x \forall y (x = y \rightarrow y = x)$.

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2. $\checkmark \exists x \neg \forall y (x = y \rightarrow y = x)$.
3. $\neg \forall y (a = y \rightarrow y = a)$. (2 $\exists$.)

3 $\checkmark \neg\forall y(a = y \rightarrow y = a)$. (2 $\exists$.)

4 $\exists y \neg(a = y \rightarrow y = a)$. (3 $\neg\forall$).

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5 $\neg(a = b \to b = a)$.

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6 $a = b$ (5 $\neg \to$)

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8 $\neg(a = a)$. $6, 7 =$. X.

- valid.

Some other equivalences (Repeated)

- $\exists x(f \vee g) \leftrightarrow \exists x f \vee \exists x g$.

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- $\forall x(f \wedge g) \leftrightarrow \forall x f \wedge \forall y g$.
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- $\exists x(f \lor g) \leftrightarrow \exists xf \lor \exists xg$.
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- $\exists y f(x_1, ..., x_n, y) \leftrightarrow \exists z f(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.

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- $\exists yf(x_1, ..., x_n, y) \leftrightarrow \exists zf(x_1, .., x_n, z)$ if neither y, z are part of $x_1, ..., x_n$.
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- But $\exists x(E(x) \wedge T(x))$ is not equivalent to $(\exists xE(x)) \wedge (\exists xT(x))$.

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