1 Introduction

About this lecture

- Ordered pairs and Cartesian products
- Relations
- More about relations
- Ordering relations
- Closures
- Equivalence relations
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- `http://plato.stanford.edu/contents.html` has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

2 Cartesian products

Cartesian products

- A, B sets. $A \times B = \{(a,b) | a \in A \wedge b \in B\}$.
- $\mathbb{R} \times \mathbb{R}$ Cartesian plane (Introduced by Descartes,.. used by Newton) First algebraic interpretation of curves...
- $P(x,y)$ The truth set of $P(x,y) = \{(a,b) \in A \times B | P(a,b)\}$.
- $x + y = 1$: $\{(a,b) \in \mathbb{R} \times \mathbb{R} | a + b = 1\}$.
- **Theorem 1.** *Suppose that A, B, C, D are sets.*
  1. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
  2. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
  3. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
  4. $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$.
  5. $A \times \emptyset = \emptyset \times A = \emptyset$.

Proof of 1

Given	Goal
A, B, C	$A \times (B \cap C) = (A \times B) \cap (A \times C)$

Given	Goal
A, B, C	$A \times (B \cap C) \subset (A \times B) \cap (A \times C)$
	$(A \times B) \cap (A \times C) \subset A \times (B \cap C)$

Proof of 1

- $\subset$ part only

Given	Goal
A, B, C	$(a, b) \in A \times (B \cap C) \to (a, b) \in (A \times B) \cap (A \times C)$
a, b	

Given	Goal
A, B, C	$(a, b) \in (A \times B) \wedge (a, b) \in (A \times C)$
$(a, b) \in A \times (B \cap C)$	

3 Relations

Example

- A and B are sets. Then $R \subset A \times B$ is a *relation* from A to B.
- Domain of R: $Dom(R) := \{a \in A | \exists b \in B((a, b) \in B)\}$.
- Range of R: $Ran(R) := \{b \in B | \exists a \in A((a, b) \in A)\}$.
- $R^{-1} := \{(b, a) \in B \times A | (a, b) \in R\}$.
- If S is a relation from B to another set C, then $S \circ R := \{(a, c) \in A \times C | \exists b \in B((a, b) \in R \wedge (b, c) \in S)\}$.

Examples

- $E = \{(c, s) \in C \times S|$ The student s is enrolled in course c $\}$.
- $E^{-1} = \{(s, c) \in S \times C|$ The course c has s as a student$\}$.
- $L = \{(s, r) \in S \times R|$ The student s lives in a room r.$\}$.
- $L^{-1} = \{(r, s) \in R \times S|$ The room r has s as tenant$\}$.
- $E \circ L^{-1} = \{(r, c) \in R \times C | \exists s \in S((r, s) \in L^{-1} \wedge (s, c) \in E)\}$.
- $= \{(r, c) \in R \times C|$ The student s lives in a room r and enrolled in course c.$\}$.
- $= \{(r, c) \in R \times C|$ Some student living in a room r is enrolled in course c.$\}$.

Theorem 2.
- $(R^{-1})^{-1} = R$.
- $Dom(R^{-1}) = Ran(R)$.
- $Ran(R^{-1}) = Dom(R)$.
- $T \circ (S \circ R) = (T \circ S) \circ R$.
- $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$. *(note order)*

Proof of 5

Given	Goal
R, S	$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Given	Goal
R, S	$(S \circ R)^{-1} \subset R^{-1} \circ S^{-1}$
	$R^{-1} \circ S^{-1} \subset (S \circ R)^{-1}$

Given	Goal
R, S	$(s, r) \in R^{-1} \circ S^{-1}$
$(s, r) \in (S \circ R)^{-1}$	

Given	Goal
R, S	$\exists t((s, t) \in S^{-1} \wedge (t, r) \in S^{-1})$
$(r, s) \in (S \circ R)$	

Proof of 5 continued

Given	Goal
R, S	$\exists t((s, t) \in S^{-1} \wedge (t, r) \in R^{-1})$
$\exists p((r, p) \in R \wedge (p, s) \in S)$	

Given	Goal
R, S	$\exists t((r, t) \in R \wedge (t, s) \in S)$
$((r, p_0) \in R \wedge (s, p_0) \in S)$	

4 More about relations

Relations as graphs

- One can draw diagrams to represent the relations: particularly when it is finite. (See Page 175 HTP).
- A relation with itself. $R \subset A \times A$.
- The identity relation $i_A = \{(x, y) \in A | x = y\}$.
- One can draw a directed graph for a relation with itself. (See P. 177 HTP).
- Example: $A = \{1, 2\}$, $B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
  - $S = \{(x, y) \in B \times B | x \subset y\}$.
  - $\{\{\emptyset, \emptyset\}, \{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}, \{\emptyset, \{1, 2\}\}, \{\{1\}, \{1\}\}, \{\{1\}, \{1, 2\}\},$
  - $\{\{2\}, \{2\}\}, \{\{2\}, \{1, 2\}\}, \{\{1, 2\}, \{1, 2\}\}\}$.

Types of self relations

- A reflexive relation: $R \subset A \times A$ is *reflexible* if $\forall x \in A(xRx)$.
- R is *symmetric* if $\forall x \forall y(xRy \to yRx)$.
- R is *transitive* if $\forall x \forall y \forall z((xRy \wedge yRz) \to xRz)$.
- $\mathbb{Z}$. $x < y$. $x \leq y$....
-

Theorem 3. *1. $R \subset A \times A$ is reflexive iff $i_A \subset R$.*

2. R is symmetric iff $R^{-1} = R$.

3. R is transitive iff $R \circ R \subset R$.