# 1 Introduction

### About this lecture

- Ordered pairs and Cartesian products
- Relations
- More about relations
- Ordering relations
- Closures
- Equivalence relations
- Course homepages: `http://mathsci.kaist.ac.kr/~schoi/logic.html` and the moodle page `http://moodle.kaist.ac.kr`
- Grading and so on in the moodle. Ask questions in moodle.

### Some helpful references

- Sets, Logic and Categories, Peter J. Cameron, Springer. Read Chapters 3,4,5.
- `http://plato.stanford.edu/contents.html` has much resource.
- Introduction to set theory, Hrbacek and Jech, CRC Press. (Chapter 2)

# 2 Ordering relations

### Ordering relations

- A relation $R \subset A \times A$ is *antisymmetric* if $\forall x \in A \forall y \in A((xRy \wedge yRx) \rightarrow y = x)$.
- $R$ is a *partial order* on $A$ if it is reflexive, transitive and antisymmetric.
- $R$ is a *total order* on $A$ if it is a partial order and $\forall x \in A \forall y \in A(xRy \vee yRx)$.

### Example

- $A = \{1, 2\}$ and $B = P(A)$.
- The subset relation is a partial order but not a total order.
- $D = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | x\text{divides } y\}$.
- $G = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \geq y\}$.

**Smallest element**

**Definition 1.** Let $R$ be a partial order on a set $A$. Let $B \subset A$ and $b \in B$.

- $b$ is called a *smallest element* of $B$ if $\forall x \in B(bRx)$.
- $b$ is *$R$-minimal* if $\neg\exists x \in B(xRb \wedge x \neq b)$.
- Which is a stronger concept?

**Example**

- $L = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \leq y\}$ which is a total order on $\mathbb{R}$. $B = \{x \in \mathbb{R} | x \geq 7\}$. $C = \{x \in \mathbb{R} | x > 7\}$.
- $L$-minimal ? $L$-smallest?
- $\mathbb{Z}^+$ with divisibility relation. $B = \{3, 4, 5, 6, 7, 8, 9\}$. $R$-minimal? $R$-smallest?
- $S = \{(x, y) \in P(\mathbb{Z}^+) \times P(\mathbb{Z}^+) | x \subset y\}$. $\mathcal{F} = \{x \in P(\mathbb{Z}^+) | 2 \in X \wedge 3 \in X\}$.
- $R$-minimal? $R$-smallest?

**Theorem 2.** *Let $R$ be a partial order on $A$. $B \subset A$.*

- *If $B$ has a smallest element, then the smallest element is unique.*
- *Suppose that b is a smallest element of $B$. Then b is minimal element of $B$ and the unique minimal element of b.*
- *If $R$ is a total order and b is a minimal element of $B$, then b is the smallest element of $B$. (not proved)*

**Proof of 1**

- | Given | Goal |
  |---|---|
  | $\exists b(\forall x \in B(bRx))$ | $\exists! b \forall x(bRx)$ |

- | Given | Goal |
  |---|---|
  | $\forall x \in B(b_0 Rx)$ | $\forall x(cRx) \rightarrow c = b_0$ |

- | Given | Goal |
  |---|---|
  | $\forall x \in B(b_0 Rx)$ | $c = b_0$ |
  | $\forall x(cRx)$ | |
  | $cRb_0, b_0 Rc$ | |

**Proof of 2**

- Divide goal. $b$ *is minimal* and $b$ is unique minimal.
-

| Given | Goal |
| --- | --- |
| $b(\forall x \in B(bRx))$ | $\neg\exists x \in B(xRb \wedge x \neq b)$ |

-

| Given | Goal |
| --- | --- |
| $(\forall x \in B(bRx))$ | $\forall x \in B\neg(xRb \wedge x \neq b)$ |

-

| Given | Goal |
| --- | --- |
| $(\forall x \in B(bRx))$ | $\forall x \in B(xRb \rightarrow x = b)$ |

-

| Given | Goal |
| --- | --- |
| $(\forall x \in B(bRx))$ | $x = b$ |
| $x \in B, xRb$ | |

**Proof of 2 continued**

- Divide goal. $b$ is minimal and $b$ *is unique minimal.*
-

| Given | Goal |
| --- | --- |
| $b(\forall x \in B(bRx))$ | $\forall c \in B((\forall x \in B(xRc \rightarrow x = c)) \rightarrow b = c)$ |

-

| Given | Goal |
| --- | --- |
| $b(\forall x \in B(bRx))$ | $b = c$ |
| $c \in B$ | |
| $\forall x \in B(xRc \rightarrow x = c))$ | |

-

| Given | Goal |
| --- | --- |
| $b(\forall x \in B(bRx))$ | $b = c$ |
| $c \in B$ | |
| $\forall x \in B(xRc \rightarrow x = c))$ | |
| $bRc,$ hence $b = c$ | |

- 
  - Largest elements: $B \subset A$. $\forall x \in B(xRb)$
  - maximal element: $\neg\exists x \in B(xRb \wedge b \neq x)$.
-

**Definition 3.** – $B \subset A$. $a$ is a *lower bound* of $B$ if $\forall x \in B(aRx)$.
- $a \in A$ is an *upper bound* of $B$ if $\forall x \in B(xRa)$.

- Let $U$ be the set of upper bounds for $B$ and let $L$ be the set of lower bounds for $B$.
- If $U$ has a smallest element, this smallest element is said to be the *least upper bound* (lub, supremum).
- If $L$ has a greatest element, this element is said to be the *greatest lower bound* (glb, infimum).

- These elements may not equal the smallest, minimal (greatest, maximal) element of $B$...

**Real number system (Hrbaceck 4.5)**

- An ordered set is *dense* if it has at least two elements and if for all $a, b \in X$, $a < b$ implies there exists $x \in X$ such that $a < x < b$.
- Let $(P, <)$ be a dense linearly ordered field. $P$ is *complete* if every nonempty subset $S$ bounded above has a supremum.

**Real number system (Hrbaceck 4.5)**

- **Theorem 4.** *Let $(P, <)$ be dense linearly ordered set without endpoints. Then there exists a complete linearly ordered set $(C, <')$ unique up to isomorphism such that*
  - *$P \subset C$. order preserved*
  - *$P$ is dense in $C$.*
  - *$C$ does not have endpoints.*
- The real number system is the completion of $\mathbb{Q}$.
- The real number system is unique complete linearly ordered set without endpoints that has a countable subset dense in it.
- Conway, Knuth invented surreal numbers...

# 3 Closures

**Reflexive closures**

**Definition 5.** • Let $R$ be a relation. The *reflexive closure* of $R$ is the smallest set $S \subset A \times A$ such that $R \subset S$ and $S$ is reflexive.

- In other words, $S$ is such that $R \subset S$, $S$ is reflexive, for every $T \subset A \times A$ and if $R \subset T$ and $T$ is reflexive, then $S \subset T$.

**Theorem 6.** *(4.5.2) Suppose that $S$ is a relation on $A$. Then $R$ has a reflexive closure.*

*Proof.* Let $S = R \cup i_A$. Properties 1, 2 are obvious. For 3, $R \subset T$. Since $T$ is reflexive, $i_A \subset T$. Thus $S = R \cup i_A \subset T$. □

**Definition 7.** Let $R$ be a relation on $A$.

- $R$ is *irreflexive* if $\forall x \in A((x,x) \notin R)$.
- $R$ is a *strict partial order* if it is irreflexive and transitive.
- $R$ is a *strict total order* if it is a strict partial order and satisfies $\forall x \in A \forall y \in A(xRy \vee yRx \vee x = y)$.

The reflexive closure of a strict partial order (resp. strict total order) is a partial order (resp. total order).

**Definition 8.** Let $R$ be a relation on $A$. The *symmetric closure* of $R$ is the smallest set $S \subset A \times A$ such that $R \subset S$ and $S$ is symmetric. This is equivalent to.

- $R \subset S$.
- $S$ is symmetric.
- For any $T \subset A \times A$ and $R \subset T$ and $T$ is symmetric imply that $S \subset T$.

**Definition 9.** Let $R$ be a relation on $A$. The *transitive closure* of $R$ is the smallest set $S \subset A \times A$ such that $R \subset S$ and $S$ is transitive. This is equivalent to.

- $R \subset S$.
- $S$ is transtive.
- For any $T \subset A \times A$ and $R \subset T$ and $T$ is transitive imply that $S \subset T$.

*Example* 10. See Figures 1,2,3 in pages 197-198 in HTP.

**Theorems**

**Theorem 11.** *Suppose that $R$ is a relation on $A$. Then $R$ has a symmetric closure.*

*Proof.* hint: $R \cup R^{-1}$. □

**Theorem 12.** *Suppose that $R$ is a relation on $A$. Then $R$ has a transitive closure.*

*Proof.* hint: Take intersections of all transitive relations containing $R$. □

# 4 Equivalence relations

### Equivalence relations

**Definition 13.** Suppose that $R$ is a relation on $A$. If $R$ is a reflexive, symmetric, and transtive, then $R$ is an *equivalence relation*.

A equivalence relation $\leftrightarrow$ a partition of a set.

**Definition 14.** Suppose that $R$ is an equivalence relation on $A$. Then the *equivalence class* of $x$ w.r.t. $R$ is $[x]_R = \{y \in A | yRx\}$.

The set of all equivalence class is denoted $A/R$ ($A$ mod $R$ )

$$A/R := \{[x]_R | x \in A\} = \{X \subset A | \exists x \in A(X = [x]_R)\}$$

### Equivalence relations

**Theorem 15.** *(4.6.5) Suppose that $R$ is an equivalence relation on $A$. Then for*

- *For all $x \in A$, $x \in [x]_R$.*
- *For all $x \in A$ and $y \in A$, $y \in [x]_R \leftrightarrow [y]_R = [x]_R$.*

**proof**

- 1. $x \in A$. Then $xRx$ by reflexivity. Thus $x \in [x]_R$.
- 2. $\rightarrow$ part:

| Given | Goal |
|---|---|
| $y \in [x]_R$ | $[y]_R = [x]_R$ |

- 

| Given | Goal |
|---|---|
| $y \in [x]_R$ | $\forall z(z \in [y]_R \leftrightarrow z \in [x]_R)$ |

- part 1:

| Given | Goal |
|---|---|
| $y \in [x]_R$ | $\forall z(z \in [y]_R \rightarrow z \in [x]_R)$ |

**proof**

- 

| Given | Goal |
|---|---|
| $y \in [x]_R$ | $zRx$ |
| $z \in [y]_R, yRx, zRy$ | |

- part 2:

| Given | Goal |
|---|---|
| $y \in [x]_R$ | $\forall z(z \in [x]_R \rightarrow z \in [y]_R)$ |

- omit

**Equivalence relation → Partition**

**Theorem 16.** *Suppose that $R$ is an equivalence relation on a set $A$. Then $A/R$ is a partition of $A$.*

*Proof.*

- To show $A/R$ is a partition of $A$, we show that $\bigcup A/R = A$, $A/R$ is pairwise disjoint, and no element of $A/R$ is empty.
- For the first item, $\bigcup A/R \subset A$. We show $A \subset \bigcup A/R$. Suppose $x \in A$. Then $x \in [x]_R$. Thus $x \in \bigcup A/R$.
- The pairwise disjointness follows from what?
- Suppose $X \in A/R$. Then $X = [x]_R \ni x$ and hence is not empty.

□

**Equivalence relation ← Partition**

**Theorem 17.** *(4.6.6) Let $A$ be a set. $\mathcal{F}$ a partition of $A$. Then there exists an equivalence relation $R$ on a set $A$ such that $\mathcal{F} = A/R$.*

We need two lemmas to prove this.

**Lemma 18.** *(4.6.7) $A$ a set. $\mathcal{F}$ a partition of $A$> Let $R = \bigcup_{X\in\mathcal{F}}(X \times X)$. Then $R$ is an equivalence relation on $A$.*

1. We call $R$ the equivalence relation induced by $\mathcal{F}$.
2. The proof is that we verify the three properties of equivalence relations.
3. We prove the transitivity: $xRy, yRz$. $(x, y) \in X \times X$ and $(y, z) \in Y \times Y$. Then $X \cap Y \ni y$. Thus, $X = Y$. Thus, $(x, z) \in X \times X$ and $xRz$.

**Lemma 19.** *(4.6.8) Let $A$ be a set. $\mathcal{F}$ a partition of $A$. Let $R$ be the equivalence relation determined by $\mathcal{F}$. Suppose $X \in \mathcal{F}$ and $x \in X$. Then $[x]_R = X$.*

- | Given | Goal |
  | --- | --- |
  | $X \in \mathcal{F}, x \in X$ | $[x]_R \subset X, X \subset [x]_R$ |

- part 1:

  | Given | Goal |
  | --- | --- |
  | $X \in \mathcal{F}, x \in X$ | $y \in X$ |
  | $y \in [x]_R$ | |

- | Given | Goal |
  | --- | --- |
  | $X \in \mathcal{F}, x \in X$ | $y \in X$ |
  | $yRx$ or $(y, x) \in Y \times Y$, Thus, $Y = X$ | |

- part 2: omit

**Proof of Theorem 4.6.6**

- Let $R = \bigcup_{X \in \mathcal{F}} X \times X$.
- We show that $A/R = \mathcal{F}$. That is, $X \in A/R \leftrightarrow X \in \mathcal{F}$.
- part 1: $\rightarrow$.

| Given | Goal |
|---|---|
| $X \in A/R$ | $X \in \mathcal{F}$ |

-

| Given | Goal |
|---|---|
| $X = [x]_R, x \in A$ | $X \in \mathcal{F}$ |
| $x \in Y$ for some $Y \in \mathcal{F}$ | |
| $Y = [x]_R$ by 4.6.8 | |
| $Y = X$ | |

**Proof of Theorem 4.6.6**

- part 2: $\leftarrow$.

| Given | Goal |
|---|---|
| $X \in \mathcal{F}$ | $X \in A/R$ |
| $X \neq \emptyset, x \in X$ | |
| $X = [x]_R \in A/R$ by 4.6.8 | |