Differentiable Manifolds Exam: Midterm (May 2000)

1. Rotate the circle $(x-2)^2+z^2=1$ on the xz-plane about the z-axis to obtain a surface Σ in the Euclidean 3-space $\mathbf{R}^3$.

(i) Find an atlas of Σ based on the rotation angles about the z-axis and the angular parameterization of the circle.

(ii) Show that Σ is a differentiable manifold using the coordinate charts from (i).

(iii) Show that Σ is diffeomorphic to $S^1 \times S^1$ where S^1 is a unit circle with the obvious differentiable structure.

2. Let M and N be smooth manifolds. Given a smooth function $f : M \to N$, show that $f^{-1}(v)$ for a regular value v of N is a smooth submanifold of M.

3. An affine motion $\mathbf{R}^n \to \mathbf{R}^n$ is a transformation of form $x \mapsto Ax + b$ where A is a nonsingular linear map, i.e., in $\mathrm{GL}(n, \mathbf{R})$, and b is a vector. An affine motion is uniquely written in this way. We define $Aff(\mathbf{R}^n)$ to be the group of all affine motions on $\mathbf{R}^n$. Show that $Aff(\mathbf{R}^n)$ is a Lie group by finding a chart.

4. Let $\mathbf{R}^3$ have the standard Euclidean coordinate functions x_1, x_2, x_3, and $\mathbf{R}^2$ have the coordinate functions x and y. A stereographic projection of the unit sphere $S^2 \subset \mathbf{R}^3$ from the north pole $N = (0, 0, 1)$ is given by $X : x = x_1/(1 - x_3), y = x_2/(1 - x_3)$. Recall the spherical coordinates (ρ, ϕ, θ) on $\mathbf{R}^3$ where $\rho = 1$ on S^2. Define

$$Y : S^2 - I \to (0, \pi) \times (-\pi, \pi) \subset \mathbf{R}^2$$

by (ϕ, θ) restricted on S^2 where I is the set of points on S^2 where $x \leq 0, y = 0$.

(i) Show that $X : S^2 - \{N\} \to \mathbf{R}^2$ is a coordinate chart, and so is Y.

(ii) Let h be a height function on S^2, i.e., $h = x_3|S^2$. Compute $\partial h/\partial x, \partial h/\partial y, \partial h/\partial \phi$, and $\partial h/\partial \theta$.

(iii) Show that two charts are C^∞ related, and write $\partial/\partial\phi$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$ at $(1, 0, 0)$.

5. Let $R_t : S^2 \to S^2$ be a rotation by angle t about the x_3-axis given by

$$R_t(x_1, x_2, x_3) = (x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, x_3).$$

(Use notations of the problem 4.) R_t is a flow generated by a vector field V on S^2, which you will find below.

(i) For each $p \in S^2$ describe the path class of the arc $\phi_t(p)$ at p using coordinate charts X and Y.

(ii) Describe the vector field generating R_t using X; that is find $X_*(V)$. Next describe it using Y.

(iii) Justify the claim that R_t is a flow generated by a vector field.