Differentiable Manifolds Exam: Midterm (April 1999)

1. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be the map given by $f(x, y) = (xy, y^2)$. Let X be a vector field on $\mathbf{R}^2$ given by $X(x, y) = (x, y)$ and ω a covariant vector field given by $(dx + dy)/(x^2 + y^2)$. Find expressions of $f_*(X)$ and $f^*(\omega)$ and justify your answers.

2. Describe by coordinates the map $\det_* : T\mathrm{GL}(2, \mathbf{R}) \to \mathbf{R}$ which is induced by the map $\det : \mathrm{GL}(2, \mathbf{R}) \to \mathbf{R}$. Find the matrix expression of $\mathbf{dev}_*$. From this, show that $\mathrm{SL}(2, \mathbf{R})$ is a Lie group. ($T\mathrm{GL}(2, \mathbf{R})$ is the tangent bundle of $\mathrm{GL}(2, \mathbf{R})$.)

3. Let A be a map $\mathbf{R}^2 \to \mathbf{R}^2$ given by $(x, y) \mapsto (x + 1, -y)$. A Möbius band M can be described as the quotient topological space

$$\{(x, y) | x \in \mathbf{R}, -1 < y < 1\}/ \sim$$

where $\sim$ is given by $(x, y) \sim (x', y')$ if $(x', y') = A^n(x, y)$ for some integer n.

(i) Give on M a C^∞-structure.

(ii) Show that

$$N = \{(x, y) | x \in \mathbf{R}, -1 < y < 1\}$$

has a C^∞ structure and covers M with the smooth covering map p.

(iii) Show that the tangent bundle TN is diffeomorphic to $N \times \mathbf{R}^2$. (An obvious chart is the inclusion $N \to \mathbf{R}^2$.)

(iv) Show that p induces a smooth map $p_* : TN \to TM$ which is a covering map. (For a component U' of $p^{-1}(U)$ such that $p|U' : U' \to U$ is a diffeomorphism, we can use chart of U given by $x \circ (p|U')^{-1}$ for a chart x of U'.)

(v) Show that $p_* \circ A^n_* = p_*$ for every n. Describe the action of the deck transformation group of the covering map $p_* : TN \to TM$ in terms of realization $TN \cong N \times \mathbf{R}^2$. (Try to figure out precisely which vectors of TN map to a same vector in TM.)

(vi) Show that TN is diffeomorphic to

$$\{(x, y; u, v) | x \in \mathbf{R}, -1 < y < 1, u, v \in \mathbf{R}\}$$

where $\sim$ is given by $(x', y'; u', v') \sim (x, y; u, v)$ if $(x', y') = A^n(x, y)$ and $(u', v') = (u, \pm v)$ where $+$ is used if n is even and $-$ otherwise.

(vii) Show that M is not orientable using (vi).