Deforming convex $\mathbb{R}P^3$-structures on 3-orbifolds

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Outline

- A $\mathbb{R}P^n$-structure (projectively flat structure; real projective structure; projective structure) on an orbifold is a geometric structure on the orbifold modelled on $(\mathbb{R}P^n, \mathrm{PGL}(n+1, \mathbb{R}))$.
- A convex $\mathbb{R}P^n$-orbifold is the quotient orbifold of a convex domain in $\mathbb{R}P^n$ by a discrete group of projective automorphisms in $\mathrm{PGL}(n+1, \mathbb{R})$. Hyperbolic 3-orbifolds form a subclass.
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- These are related to affine manifolds and conformal manifolds.
- The study of convex $\mathbb{R}P^n$-structures on manifolds were begun by Cartan (1930s), Kuiper, Koszul, Benzecri, and Vey (1960-1970s) They discovered "openness" of $\mathbb{R}P^n$-structures and the properties of the convex domains.
- Real projective structures on surfaces were studied by Sullivan, Thurston, (1980s), Goldman, Choi: Construction, classification, convex decomposition, closedness. (1990s)

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- The convex $\mathbb{R}P^3$-manifolds were begun to be studied by Cooper, Long, and Thistlethwaite [2], [3]. They showed about 5 percent out of 5000 deforms nontrivially (more than bending or conformal cases.)
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- Coxeter 3-orbifolds were begun to be studied by Benoist and myself. Orderability and deformations spaces.
- L. Marquis [4] completed one subcase: iterated truncation tetrahderal Coxeter 3-orbifolds related to orderable orbifolds
- Choi, Hodgson, Lee [1] did a numerical study of $\mathbb{R}P^3$-structures on Coxeter orbifolds. Open ideal or hyperideal hyperbolic Coxter 3-orbifolds.

Outline

- Now to nonreflection type orbifolds with ends: (Tillman's example)
- Open and closedness of the deformation space of convex $\mathbb{R}P^n$-structures on orbifolds with radial ends
- The relative hyperbolicity and strict convexity for orbifolds with radial ends. (Note that Cooper and Tillman have similar results independently.)

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- The relative hyperbolicity and strict convexity for orbifolds with radial ends. (Note that Cooper and Tillman have similar results independently.)
- Finally, we discuss open problems in this area: Deformability versus rigidity, noncompactness of the deformation spaces.
- Maybe algebraic geometry techniques, harmonic bundle theory could help...
- I won't be covering the recent results of Huesener and Porti– cohomological approach

Orbifolds

- By an n-dimensional orbifold, we mean a Hausdorff second countable topological space with a fine open cover $\{U_i, i \in I\}$ with models $(\tilde{U}_i, G_i)$ where G_i is a finite group acting on the open subset $\tilde{U}_i$ of $\mathbb{R}^n$ and a map $p_i : \tilde{U}_i \to U_i$ inducing homeomorphism $\tilde{U}_i/G_i \to U_i$ where
- for each i, j, $x \in U_i \cap U_j$, there exists U_k with $x \in U_k \subset U_i \cap U_j$.
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- A $\mathbb{R}P^n$-structure on an orbifold is given by having charts from U_is to open subsets of $\mathbb{R}P^n$ with transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$.
- A good orbifold: M/Γ where Γ is a discrete group with a properly discontinuous action.

Real projective structures on orbifolds

- Suppose that a discrete group Γ act on a manifold M properly discontinuously.
- A $\mathbb{R}P^n$-structure on M/Γ with simply connected M is given by an immersion $D : M \to \mathbb{R}P^n$ equivariant with respect to a homomorphism $h : \Gamma \to \mathrm{PGL}(n+1, \mathbb{R})$ where Γ is the fundamental group of M/Γ.
- For example, let M be an interior of a conic in $\mathbb{R}P^n$. Then $\Gamma \subset PO(n, 1)$ and M/Γ is a hyperbolic orbifold and a convex $\mathbb{R}P^n$-orbifold. We can deform these to nonhyperbolic but convex $\mathbb{R}P^n$-orbifolds sometimes (Kac-Vinberg ...). – The subject of this talk.

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- A $\mathbb{R}P^n$-structure on M/Γ is *convex* if $D(M)$ is a convex domain in an affine subspace $A^n \subset \mathbb{R}P^n$. In this case, we will identify M with $D(M)$ for a particular choice of D and Γ with its image under h.
- A *properly convex* domain is a convex domain that is a precompact domain in some affine subspace. A convex domain is properly convex iff it does not contain a complete real line.

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- A $\mathbb{R}P^n$-structure on M/Γ is *properly convex* if so is $D(M)$.

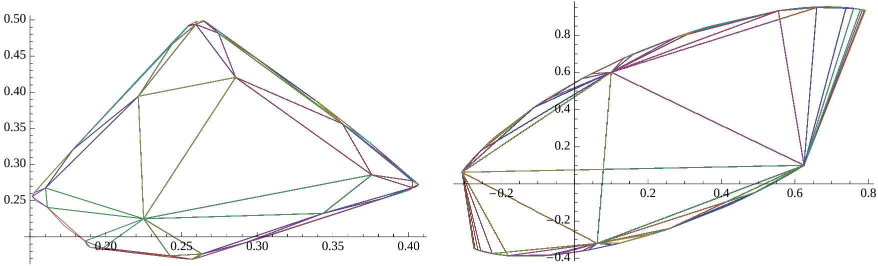

Figure: The developing images of convex $\mathbb{R}P^n$-structures on 2-orbifolds deformed from hyperbolic ones: $S^2(3, 3, 5)$ and $D^2(2, 7)$

Motivations to study $\mathbb{R}P^n$-structures

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- Orbifolds with convex $\mathbb{R}P^n$-structures have Hilbert metrics with many properties of $CAT(0)$-spaces and the theory is compatible with the geometric group theory. (no angles...) (N. Kuiper, Benzecri, Colbois, Venicos, Verovic,..)

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- There are "many" more orbifolds with $\mathbb{R}P^n$-structures than homogeneous Riemannian ones.

Motivations to study $\mathbb{R}P^3$-structures on 3-orbifolds

- Real projective structures on surfaces and 2-orbifolds are "understood":
  - There is a constructive classification from the convex decomposition theorem and the annulus decomposition theorem:
  - There is always a decomposition into convex subsurfaces and annuli. Convex subsurfaces and annuli with geodesic boundary are classifiable. (Sullivan-Thurston, Goldman, Choi)

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- There is a numerical study by Cooper, Long, and Thistlethwaite on the $\mathbb{R}P^3$-structures obtained by deforming hyperbolic 3-manifolds in the Hodgson-Weeks census of 5000. About 5 percents are deformable. Some of the computations are exact and includes many interesting examples.
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- Question (CLT): Does every hyperbolic 3-orbifold deform up to finite covers?

Deformation spaces of convex $\mathbb{R}P^n$-structures

- Given an orbifold S, a *convex $\mathbb{R}P^n$-structure* is given by a diffeomorphism $f : S \to \Omega/\Gamma$ for a convex domain Ω in $\mathbb{R}P^n$ and Γ a subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$.
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- The *deformation space* $\mathrm{CDef}(S)$ *of convex $\mathbb{R}P^n$-structures* is $\{(D,h)\}/\sim$ where $(D,h) \sim (D',h')$ if there is an isotopy $\tilde{f} : \tilde{S} \to \tilde{S}$ such that $D' = D \circ \tilde{f}$ and $h'(\tilde{f}g\tilde{f}^{-1}) = h(g)$ for each $g \in \pi_1(S)$ or $D' = k \circ D$ and $h'(\cdot) = kh(\cdot)k^{-1}$ for $k \in \mathrm{PGL}(n+1,\mathbb{R})$.

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- Alternatively, $\mathrm{CDef}(S) = \{f : S \to \Omega/\Gamma\}/\sim$ where $f \sim g$ for $f : S \to \Omega/\Gamma$ and $g : S \to \Omega'/\Gamma'$ if there exists a projective diffeomorphism $k : \Omega/\Gamma \to \Omega'/\Gamma'$ so that $k \circ f$ is homotopic to g.

The hol map: The local homeomorphism property

The closed version is a classical theorem due to Koszul that the holonomy representations locally parametrize the geometric structures and vice versa. We state the radial end version.

Theorem A

Let $\mathcal{O}$ be a closed n-orbifold (or noncompact tame with radial ends), (suppose that $\mathcal{O}$ has the end fundamental group conditions.) Then the following map is a local homeomorphism:

$$\mathrm{hol} : \mathrm{Def}_{(E)}(\mathcal{O}) \to \mathrm{rep}_{(E)}(\pi_1(O), \mathrm{PGL}(n+1, \mathbb{R})).$$

Proof.

This follows as in the compact cases using the bump functions. However, we may need the bump functions extending to the ends for radial ends.

Convexity.

- We begin by discussing the convexity:
- The usual version is for closed orbifolds.

Proposition

- *A $\mathbb{R}P^n$-orbifold (with nonempty radial end) is convex if and only if the developing map sends the universal cover to a convex open domain in $\mathbb{R}P^n$.*
- *A $\mathbb{R}P^n$-orbifold (with nonempty radial end) is properly convex if and only if the developing map sends the universal cover to a properly convex open domain in a compact domain in an affine patch of $\mathbb{R}P^n$.*
- *If a convex $\mathbb{R}P^n$-orbifold (with nonempty radial end) is not properly convex, then its holonomy is reducible.*

Benoist's "maximally complete" results

Benoist in his papers "Convexes divisibles I-IV":

Proposition (Benoist)

Suppose that a discrete subgroup Γ of $\mathrm{PGL}(n+1, \mathbb{R})$ acts on a properly convex n-dimensional open domain Ω so that Ω/Γ is compact. Then the following statements are equivalent.

- *Every finite index subgroup of Γ has a finite center.*
- *Every FI subgroup of Γ has a trivial center.*
- *Every FI subgroup of Γ is irreducible in $\mathrm{PGL}(n+1, \mathbb{R})$. (or strongly irreducible).*
- *The Zariski closure of Γ is semisimple.*
- *Γ does not contain a normal infinite nilpotent subgroup.*
- *Γ does not contain a normal infinite abelian subgroup.*

Benoist's result continued

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- Theorem (Benoist)

  Let Γ *be a discrete subgroup of* $\mathrm{PGL}(n+1,\mathbb{R})$ *with a trivial virtual center. Suppose that a discrete subgroup* Γ *of* $\mathrm{PGL}(n+1,\mathbb{R})$ *acts on a properly convex n-dimensional open domain* Ω *so that* Ω/Γ *is compact. Then every representation of a component of* $\mathrm{Hom}(\Gamma, \mathrm{PGL}(n+1,\mathbb{R}))$ *containing the inclusion representation also acts on a properly convex n-dimensional open domain cocompactly.*

Coxeter 3-orbifolds

We will concentrate on 3-dimensional orbifolds whose base spaces are convex polyhedra and whose sides are silvered and each edge is given an order. For example: a hyperbolic polyhedron with edge angles of form π/m for positive integers m.

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The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots, f, (R_i R_j)^{n_{ij}} = 1$$

where R_i is associated with silvered sides and $R_i R_j$ has order n_{ij} associated with the edge formed by the i-th and j-th side meeting.

Coxeter orbifold structure

Let P be a fixed convex polyhedron. Let us assign orders at each edge. We let e be the number of edges and e_2 be the numbers of edges of order-two. Let f be the number of sides.

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We keep vertices of P of form $(2, 2, n), n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5),$ i.e., orders of spherical triangular groups and remove others. This makes P into an open 3-dimensional orbifold with ends.

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Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from P with vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

Vinberg's results...

- Vinberg studied these as linear reflection goups. His main results is that a closed $\mathbb{R}P^n$-orbifold $\hat{P}$ is properly convex, i.e., $\hat{P}$ is a quotient of a precompact convex domain in an affine subspace of $\mathbb{R}P^n$.
- A linear reflection group is determined by the polytope given by equations $a_i \equiv 0$ for $i = 1, .., f$ and the reflection points b_i, $i = 1, .., f$.

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- A linear reflection group is determined by the polytope given by equations $a_i \equiv 0$ for $i = 1, .., f$ and the reflection points b_i, $i = 1, .., f$.
- Cartan matrix: $(a_{ij} = a_i(b_j))_{ij}$ satisfies
  - $a_{ij} \leq 0, i \neq j$, and if $a_{ij} = 0$, then $a_{ji} = 0$.
  - $a_{ii} = 2, a_{ij}a_{ji} \geq 4$, or $a_{ij}a_{ji} = 4\cos^2(\pi/n_{ij})$.
- In general, symmetric Cartan matrices can be deformed to nonsymmetric Cartan matrices $(a_{ij} = a_i(b_j))_{ij}$ and they correspond to the deformations.

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- In general, symmetric Cartan matrices can be deformed to nonsymmetric Cartan matrices $(a_{ij} = a_i(b_j))_{ij}$ and they correspond to the deformations.
- The rank of the matrix equals one + the dimension of the Coxeter orbifold. The cyclic invariants $a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_k i_1}$ for distinct indices are complete invariants.
- Kac and Vinberg found examples of convex $\mathbb{R}P^n$-orbifolds that are not Riemannian hyperbolic based on hyperbolic reflection triangle groups and deforming.

Deformation spaces

- The *deformation space* $D(\hat{P})$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.
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- A point p of $D(\hat{P})$ always determines a fundamental polyhedron P up to projective automorphisms.
- We wish to understand the space where the fundamental polyhedron is always projectively equivalent to P.
- We call this the *restricted deformation space* of $\hat{P}$ and denote it by $D_P(\hat{P})$.

Orderable Coxeter 3-orbifolds

We say that the polytope P is *orderable* if we can order the sides of P so that each side meets sides of higher index in less than or equal to 3 edges.

Definition

Let $\hat{P}$ be the orbifold obtained from P by silvering sides and removing vertices as above. We also say that the orbifold $\hat{P}$ is *orderable* if the sides of P can be ordered so that each side has no more than three edges which are either of order 2 or included in a side of higher index.

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Theorem

Let P be a convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P)$ be the dimension of the group of projective automorphisms acting on P. Suppose that $\hat{P}$ is orderable. Then the restricted deformation space of projective structures on the orbifold $\hat{P}$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.

Proof.

The basic idea is to control the reflection points in order. Basically, this is the "underdetermined case" in terms of algebraic equations. (Others are usually "overdetermined cases".) □

The total deformation space fibers over the open subspace of polytopes combinatorially equivalent to P.

Iterated-truncation tetrahedron (ecimaedre combinatoire)

Theorem of L. Marquis

We start with a tetrahedron and cut off a vertex. We iterate this. This gives us a convex polytope with trivalent vertices. Then the deformation space is diffeomorphic to $\mathbb{R}^{e_+ - 3}$ when the orbifold satisfies Andreev's conditions.

The notion is more general than that of orderable, compact hyperbolic types. The proof is basically very combinatorial and algebraic over $\mathbb{R}$. (generalizations?)

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Choudhury, Lee, Choi

In fact, *OCH* are only five types: tetrahedron, prism, and three other. There are infinitely many orderable noncompact Coxeter 3-orbifolds admitting hyperbolic structures.

Hyperbolic ideal (or hyperideal) Coxeter 3-orbifolds

- Now we are interested in nonorderable cases and some overdetermined cases as well.
- For ideal or hyperideal hyperbolic Coxeter 3-orbifolds with all edge orders ≥ 3, Lee, Hodgson, and Choi showed that the restricted deformation space of convex $\mathbb{R}P^3$-structures is locally a smooth cell of dimension 6 at the hyperbolic point.
- The deformation space has dimension $e - 3$ and smooth at the hyperbolic point.
- The proof involves Weil-Prasad infinitesimal rigidity:

The equations to solve

- We now fix orders n_{ij} for the codimension 2 faces of P and consider the *restricted* deformation space of the corresponding Coxeter orbifold $\hat{P}$. Now the α_i's will be *fixed*, and b_i's are *variables* ; Vinberg's result leads us to solve the following system of polynomial equations:

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  - For each $i = 1, \dots, f$,
  $$a_{ii} = \alpha_i(b_i) = 2, \tag{1}$$
  - If F_i and F_j are adjacent in P and $n_{ij} > 2$,
  $$a_{ij}a_{ji} = \alpha_i(b_j)\alpha_j(b_i) = 4\cos^2(\pi/n_{ij}), \tag{2}$$
  - If F_i and F_j are adjacent in P and $n_{ij} = 2$,
  $$a_{ij} = \alpha_i(b_j) = 0 \text{ and } a_{ji} = \alpha_j(b_i) = 0. \tag{3}$$

The equations to solve

- We now fix orders n_{ij} for the codimension 2 faces of P and consider the *restricted* deformation space of the corresponding Coxeter orbifold $\hat{P}$. Now the α_i's will be *fixed*, and b_i's are *variables* ; Vinberg's result leads us to solve the following system of polynomial equations:
  - For each $i = 1, \ldots, f$,
  $$a_{ii} = \alpha_i(b_i) = 2, \tag{1}$$
  - If F_i and F_j are adjacent in P and $n_{ij} > 2$,
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  $$a_{ij} = \alpha_i(b_j) = 0 \text{ and } a_{ji} = \alpha_j(b_i) = 0. \tag{3}$$
- Note the difference between the cases $n_{ij} = 2$ and $n_{ij} > 2$.
- If $n_{ij} > 2$ always, then actually with α_i's fixed, the differential coincide with the differential for solving for hyperbolic structures where a_i is Lorentz dual to b_i.

Numerical experiments on cubes and dodecahedrons

Following up on the Cooper-Long-Thistlethwaite approach, Choi, Hodgson, Lee showed

Theorem

Consider the compact hyperbolic cubes such that each dihedral angle is $\pi/2$ or $\pi/3$. Up to symmetries, there exist 34 cubes satisfying this condition. For the corresponding hyperbolic Coxeter orbifolds,

- *10 are projectively deformable relative to the mirrors*
- *and the remaining 24 are projectively rigid relative to the mirrors.*
- *The deformations of the three orbifolds are not projective bendings.*

Some of these with many 2s are shown to be rigid by "a linear test". We use computations packages and some of these need Gröbner basis techniques.

The cubes

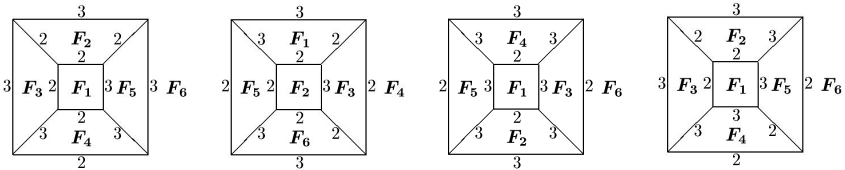

Figure: Some of the cubes we studied: cu15, cu21, cu33, cu34

Notation

- Each e_i is an edge order, corresponding to a dihedral angle π/e_i,
- O = the number of variables $-$ the number of Vinberg equations,
- I = dim of infinitesimal restricted deformation space of rp structures,

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- L = Is it possible to apply the linear test of rigidity? (yes or no), and the maximum level needed,
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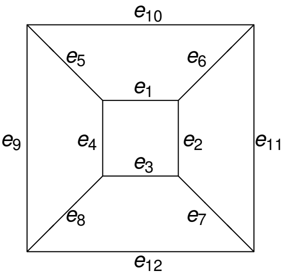
- S = min of the singular values of the Jacobian D.
- G = order of the group of symmetries,
- C = number of (essential) circuits in the dual graph consisting of edges of order 3
- B = number of totally geodesic 2-dimensional suborbifold (nonfacial)

Labels of edges of cubes



The some of results

name	$e_1 e_2 \cdots e_{11} e_{12}$	O	I	A	L	J	G	C	B
cu1	232222232223	-3	0	0	yes, level 2	·	**2**	**0**	0
cu2	232222232233	-2	0	0	yes, level 3	·	**1**	**0**	0
cu3	232222232322	-3	0	0	yes, level 1	·	**6**	**0**	0
cu4	232222232323	-2	0	0	yes, level 2	·	**1**	**0**	0
cu5	232222232333	-1	0	0	yes, level 3	·	**1**	**0**	0
cu6	232222233322	-2	0	0	yes, level 2	·	**2**	**0**	0
cu7	232222233332	-1	0	0	yes, level 3	·	**1**	**0**	0
cu8	232222322223	-3	0	0	yes, level 2	·	**2**	**0**	0
cu9	232222322332	-2	0	0	yes, level 2	·	**1**	**0**	0
cu10	232222323223	-2	0	0	yes, level 3	·	**2**	**0**	0
cu11	232222323322	-2	0	0	yes, level 2	·	**2**	**0**	0
cu12	232222323323	-1	0	0	yes, level 3	·	**1**	**0**	0
cu13	232222323332	-1	0	0	yes, level 2	·	**2**	**0**	0
cu14	232222333322	-1	0	0	yes, level 3	·	**1**	**0**	0

Table: The list of cubes up to cu14

See `http://mathsci.kaist.ac.kr/~manifold/cudo.zip` for the computation files.

name	$e_1 e_2 \cdots e_{11} e_{12}$	O	I	A	L	J	**G**	**C**	**B**
cu15	232222333332	0	0	0	no	yes	**1**	**1**	**0**
cu16	232223233322	-1	0	0	yes, level 3	·	**2**	**0**	0
cu17	232223322323	-1	1	**1**	no	no	**2**	**1**	**1**
cu18	232223323323	0	1	**1**	no	no	**4**	**1**	**1**
cu19	232223333322	0	0	0	no	yes	**1**	**1**	**0**
cu20	232232232233	-1	0	0	yes, level 3	·	**2**	**0**	0
cu21	232232232323	-1	1	**1**	no	no	**4**	**1**	**1**
cu22	232232232333	0	1	**1**	no	no	**2**	**1**	**1**
cu23	232232332322	-1	0	0	yes, level 3	·	**2**	**0**	0
cu24	232232332323	0	0	0	no	yes	**1**	**1**	**0**
cu25	232232332332	0	0	0	no	yes	**1**	**1**	**0**
cu26	232233332223	0	1	**0**	no	no	**2**	**1**	**0**
cu27	232233332323	1	2	**1**	no	no	**2**	**2**	**1**
cu28	232322232233	-1	0	0	no	yes	**2**	**1**	**0**
cu29	232323232323	0	1	**0**	no	no	**4**	**1**	**2**
cu30	232323323323	1	1	1	no	yes	**4**	**2**	**1**
cu31	232323332323	1	1	1	no	yes	**2**	**2**	**0**
cu32	232323333322	1	1	1	no	yes	**2**	**2**	**0**
cu33	232333332323	2	3	**2**	no	no	**8**	**3**	**2**
cu34	233223233322	0	1	**1**	no	no	**12**	**1**	**3**

Table: The list of cubes

Discussions

- If L and J are no, we use Gröbner basis to find exact solutions.
- cu31, cu32, and cu34 are only orbifolds here with deformations relative to mirrors that are not projective bendings.
- cu17, cu18, cu21, cu22, cu27, cu30 have 1-dimensional deformations relative to mirrors that are projective bendings.

Discussions

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- cu31, cu32, and cu34 are only orbifolds here with deformations relative to mirrors that are not projective bendings.
- cu17, cu18, cu21, cu22, cu27, cu30 have 1-dimensional deformations relative to mirrors that are projective bendings.
- For cu33, there are two parameters of projective bendings giving a 2-dimensional space of bendings by Theorem 5.3 of Johnson-Millson [3]. The deformation space is singular according to Lemma 6.1 in [3]. The Gröbner basis has a squared term.
- For cu29 and cu34, the totally geodesic suborbifolds do not yield any projective bendings.

Dodecahedrons

Theorem

Consider the compact hyperbolic dodecahedra such that each dihedral angle is $\pi/2$ or $\pi/3$, and each face has at most two dihedral angles equal to $\pi/2$.

- *Up to symmetries, there exist 13 dodecahedra satisfying these conditions.*
- *For the corresponding hyperbolic Coxeter orbifolds, only 1 has projective deformations relative to the mirrors, which are not projective bendings, and*
- *the remaining 12 are projectively rigid relative to the mirrors.*

Dodecahedron

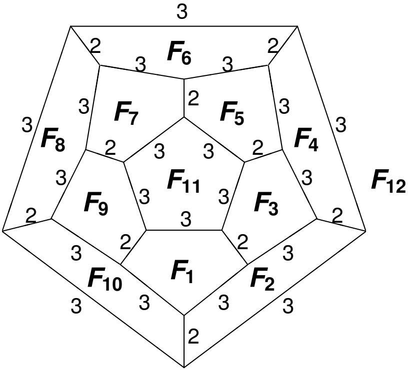


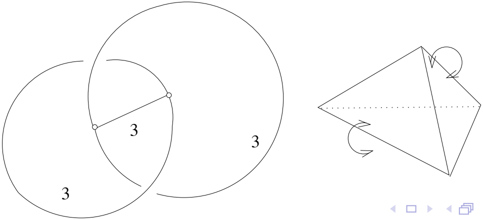
Figure: do13 with five-fold rotational symmetry about the axis.

This is the dodecahedral orbifold with a 1-dimensional restricted deformation space that we found. It is orderable up to rotational symmetry. In fact, we have deformability of orderable polytopes up to rotational symmetry.

Clearly we need to work out more examples to figure out what is the precise condition where there are deformations which are bendings or not. More theoretical approach is called for.

S. Tillman's example

- There is a census of small hyperbolic orbifolds with graph-singularity. (See the paper by D. Heard, C. Hodgson, B. Martelli, and C. Petronio [2])
- There is a complete hyperbolic structure on the orbifold based on $\mathbf{S}^3$ with handcuff singularity with two points removed. The singularity orders are three. This is obtained by gluing a pair of faces of a tetrahedron around a pair of disjoint edges.



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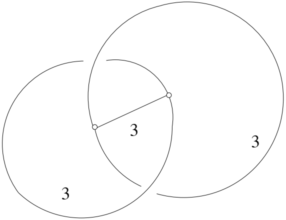
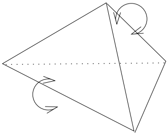
- There is a one-parameter space of deformations of the structures to $\mathbb{R}P^3$-structures as seen by simple matrix computations.
- These are all properly and strictly convex and irreducible by our theory to be presented.





End orbifold

- A $\mathbb{R}P^n$-orbifold has *radial ends* if each end has an end neighborhood foliated by concurrent geodesics for each chart ending at the common point of concurrency.
- Each end has a neighborhood diffeomorphic to a closed orbifold times an open interval.

End orbifold

- A $\mathbb{R}P^n$-orbifold has *radial ends* if each end has an end neighborhood foliated by concurrent geodesics for each chart ending at the common point of concurrency.
- Each end has a neighborhood diffeomorphic to a closed orbifold times an open interval.
- Given an end, there is an *end orbifold* associated with the end. The radial foliation has a transversal $\mathbb{R}P^{n-1}$-structure and hence the end orbifold has an induced $\mathbb{R}P^{n-1}$-structure of one dimension lower.
- The end orbifold is convex if $\mathcal{O}$ is convex. If the end orbifold is properly convex, then we say that the end is a *transversely properly convex end*.

Classification of ends: rather restrictions on ends

- A subdomain K of $\mathbb{R}P^n$ is said to be *horospherical* if it is strictly convex and the boundary ∂K is diffeomorphic to $\mathbb{R}^{n-1}$ and $\mathrm{bd}K - \partial K$ is a single point.
- K is *lens-shaped* if it is a convex domain and ∂K is a disjoint union of two smoothly embedded $(n-1)$-cells not containing any straight segment in them.
- A *cone* is a domain in $\mathbb{R}P^n$ whose closure in $\mathbb{R}P^n$ has a point in the boundary, called a *cone-point*, so that every other point has a segment contained in the domain with endpoint the cone point and itself.

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- A *cone-over* a lens-shaped domain A is a convex submanifold that contains a lens-shaped domain A of the same dimension and
  - is a union of segments from a cone-point $v \notin A$ to points of A,
  - the manifold boundary is one of the two boundary components of A, and
  - each maximal segment from v meets the two boundary components at unique points.

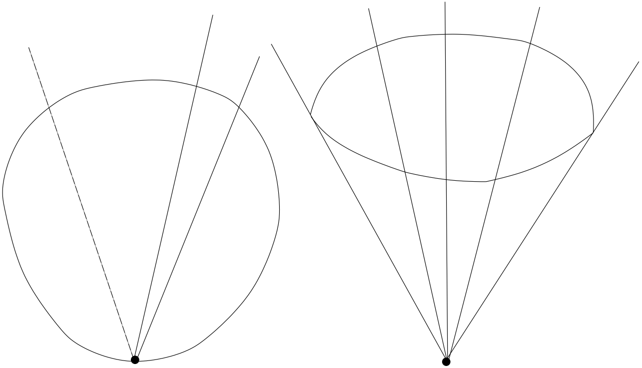

Figure: The universal covers of horospherical and lens shaped ends. The radial lines form cone-structures.

- A *lens-cone* is the union of the segments over a lens-shaped domain.
- A *lens* is the lens-shaped domain A, not determined uniquely by the lens-cone itself.

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- A *cone-over* a totally-geodesic domain A is a cone over a point x not in the hyperspace.
- An admissible ends are ones modeled on above (or joins) with the fundamental group a virtual product of abelian and hyperbolic groups. (strictly for convenience)

Openness

Theorem B

Let $\mathcal{O}$ be a noncompact topologically tame n-orbifold with admissible ends. Suppose that $\mathcal{O}$ satisfies the convex end fundamental group conditions. Then

- *In $\mathrm{Def}^i_{E,ce}(\mathcal{O})$, the subspace $\mathrm{CDef}_E(\mathcal{O})$ of IPC-structures is open.*
- *Suppose further that $\pi_1(\mathcal{O})$ contains no notrivial nilpotent normal subgroup. The deformation space $\mathrm{CDef}_{E,ce}(\mathcal{O})$ of IPC-structures on $\mathcal{O}$ maps homeomorphic to a component of $\mathrm{rep}^i_{E,ce}(\pi_1(\mathcal{O}), \mathrm{PGL}(n+1, \mathbb{R}))$.*

Theorem C

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Let $\mathcal{O}$ be a strict IPC noncompact topologically tame n-dimensional orbifold with admissible ends and convex end fundamental group condition. Suppose also that $\mathcal{O}$ has no essential homotopy annulus or torus. Then

- $\pi_1(\mathcal{O})$ *is relatively hyperbolic with respect to its end fundamental groups.*

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- *In* $\mathrm{Def}^i_{E,ce}(\mathcal{O})$, *the subspace* $\mathrm{SDef}^i_E(\mathcal{O})$ *of strict IPC-structures with respect to the ends is open.*
- *The deformation space* $\mathrm{SDef}_{E,ce}(\mathcal{O})$ *of strict IPC-structures on* $\mathcal{O}$ *with respect to the ends maps homeomorphic to a component of*

$$\mathrm{rep}^i_{E,ce}(\pi_1(\mathcal{O}), \mathrm{PGL}(n+1, \mathbb{R})).$$

Hilbert metrics

- A *Hilbert metric* on an IPC-structure is defined as a distance metric given by cross ratios. (We do not assume strictness here.)
- Let Ω be a properly convex domain. Then $d_\Omega(p, q) = \log(o, s, q, p)$ where o and s are endpoints of the maximal segment in Ω containing p, q.

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- Let Ω be a properly convex domain. Then $d_\Omega(p, q) = \log(o, s, q, p)$ where o and s are endpoints of the maximal segment in Ω containing p, q.
- This gives us a well-defined Finsler metric.
- Given an IPC-structure on $\mathcal{O}$, there is a Hilbert metric d_H on $\tilde{\mathcal{O}}$ and hence on $\tilde{\mathcal{O}}$. This induces a metric on $\mathcal{O}$.

Relatively hyperbolicity and stric IPC-structures

- We will use Bowditch's result to show

Theorem (D)

Let $\mathcal{O}$ be a topologically tame strictly IPC-orbifold with radial ends and has no essential annuli or tori. Then $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to the end groups $\pi_1(E_1), ..., \pi_1(E_k)$.

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- Fact: Suppose that $\pi_1(E_l), .., \pi_1(E_k)$ are hyperbolic for some $0 \leq l < k$, $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to $\pi_1(E_1), \ldots, \pi_1(E_{l-1})$ iff so it is with respect to $\pi_1(E_1), \ldots, \pi_1(E_k)$. (Drutu)

- Proof: We denote this quotient space $\mathrm{bd}\tilde{\mathcal{O}}_1/\sim$ by B.
- We will use Theorem 0.1. of Yaman [5]: We show that $\pi_1(\mathcal{O})$ acts on the set B as a geometrically finite convergence group.

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- The group acts properly discontinuously on the set of triples in B.
- An end group Γ_x for end vertex x is a parabolic subgroup fixing x since the elements in Γ_x fixes only the contracted set in B and since there are no essential annuli.

- Proof continued: Let p be a point that is not a horospherical endpoint or a singleton corresponding an lens-shaped end. We show that p is a conical limit point.

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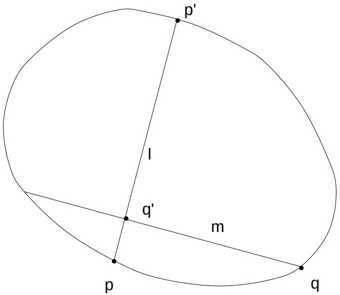

Figure: A shortest geodesic m to a geodesic l.

Converse

We will prove the partial converse to the above Theorem D:

Theorem (E)

Let $\mathcal{O}$ be a topologically tame IPC-orbifold with admissible ends without essential annuli or tori. Suppose that $\pi_1(\mathcal{O})$ is relatively hyperbolic group with respect to the admissible end groups $\pi_1(E_1), ..., \pi_1(E_k)$ where E_i are horospherical for $i = 1, ..., m$ and lens-shaped for $i = m+1, ..., k$ for $0 \leq m \leq k$.

- *Assume that $\mathcal{O}$ is IPC. Then $\mathcal{O}$ is strictly IPC.*
- *Let $\mathcal{O}_1$ be obtained by removing the concave neighborhoods of hyperbolic ends. Then if $\mathcal{O}$ is IPC, then $\mathcal{O}_1$ is strictly IPC.*

Proof.

Suppose not. We obtain a triangle T with ∂T in $\partial\tilde{\mathcal{O}}_1$. □

Lemma

Suppose that $\mathcal{O}$ is a topologically tame properly convex n-orbifold with radial ends that are properly convex or horospherical and $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to its ends. $\mathcal{O}$ has no essential tori or essential annuli. Then every triangle T in $\tilde{\mathcal{O}}$ with $\partial T \subset \partial\tilde{\mathcal{O}}$ is contained in the closure of a convex hull of its end.

Proof.

Uses asymptotic cone in Drutu-Sapir's work. □

Proofs of Theorem B and C

- We show that a small change of the structure implies the small change of the universal covers of the end orbifolds in the Hausdorff metrics.– We can control the ends.
- We show that the Koszul-Vinberg function can be perturbed to positive definite functions in the affine suspensions by controlling the ends.– This proves Theorem B.

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- We show that the Koszul-Vinberg function can be perturbed to positive definite functions in the affine suspensions by controlling the ends.– This proves Theorem B.
- For theorem C, we use "Strict IPC iff rel. hyperbolic".
- As we deform a strict IPC structure, we do not change the rel. hyperbolicity. Thus, strict IPC property is preserved.

Open problems for Coxeter orbifolds (with Hodgson and Lee)

Q. 1. Cooper, Hodgson

Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. What precise combinatorial condition tell us it is deformable or not?

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Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. What precise combinatorial condition tell us it is deformable or not?

- Linear test,
- "weakly orderability" that shows smoothness and the dimension at hyperbolic points (Lee).
- Related to symmetry or essential suborbifolds (bending) or the dual edge circuit property?

Q. 2: Solutions at infinity

Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. What is the solution at infinity in $D_P(\hat{P})$? For example, $\emptyset$ or not?

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To answer the question 2, we try to find how to compactify the solution space $D_P(\hat{P})$ using tropical metheods and valuations.

Q. 3

What is the global structure of the deformation spaces?

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- Is it noncompact always; or is there a compact deformation space? (due to Benoist, Hodgson)

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- Is it noncompact always; or is there a compact deformation space? (due to Benoist, Hodgson)

Q. 4 (Hodgson)

Let P be a 3-dimensional hyperbolic Coxeter polyhedron, and let $\hat{P}$ denote its Coxeter orbifold structure. Suppose that $n_e \geq 3$ is an order of edge e. Is the dimension of $D_P(\hat{P})$ constant when we change n_e into the different values ≥ 3?

No sufficient experimentations yet...Order increases the degree of polynomial equations and hence more difficult.

Q. 5. Projective Andreev's theorem? Goldman, Choi

What is a projective version to the Andreev's theorem for 3-dimensional hyperbolic polyhedron? Suppose that C is an abstract 3-dimensional polyhedron and orders $n_{ij} \geq 2$ are given corresponding to each edge $F_{ij} = F_i \cap F_j$ of C, where F_i are the faces of C. Which conditions are necessary and sufficient for the existence of a compact 3-dimensional "projective" polyhedron P which realizes C with "dihedral angles" π/n_{ij} at each edge F_{ij}?

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A cusp-opening is a behavior of ideal boundary becoming a totally geodesic boundary component of dim 2. This was first observed by Benoist and numerically by Lee and Choi. (Maybe there are more general behaviors..)

Q. 6. Cusp openning

Does $\hat{P}$ have cusp openings to totally geodesic boundary at some of the ideal vertices of P? In fact, we can ask this for any hyperbolic 3-manifolds with cusps. (The cusp opening seems to depend on $\hat{P}$ and on themselves. But how?)

Q. 7. Projective Dehn surgery or cone-angle deformations

- Finally, we think that we can do an $\mathbb{R}P^n$-Dehn-surgery:
- That is, given an $\mathbb{R}P^4$-manifold with radial end diffeomorphic to $T^3 \times \mathbb{R}$, we obtain a closed 4-manifolds by attaching $T^2 \times D^2$.

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- In fact we can do cone-angle deformation from 0 to ϵ, $\epsilon > 0$, where the singularity is at $T^2 \times (0,0)$ and $T^2 \times D - \{(0,0)\}$ is smooth.
- An example is obtained from simple computations of a 4-dimensional Coxeter orbifold that is a prism times $[0,1)$. (Benoist, Lee, Choi).

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- For which subset of $\mathbb{Z}^3$, the Dehn surgeries are possible from a complete hyperbolic 4-manifold to obtain $\mathbb{R}P^4$-structures on closed 4-manifolds. (convex, 2-convex, or not convex)

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