Margulis space-time with parabolics

Suhyoung Choi (with Drumm, Goldman)

KAIST

July, 2018

Outline

Outline

Article: arXiv:1710.09162

Margulis space-times

- **Isom**$^+$(E) the group of Lorentzian isometries on the flat Lorentzian space E.

Margulis space-times

- $\mathbf{Isom}^{+}(\mathbb{E})$ the group of Lorentzian isometries on the flat Lorentzian space $\mathbb{E}$.
- A discrete affine group Γ acting properly on $\mathbb{E}$ is either solvable or is free of rank ≥ 2.
- Γ is a *proper affine free group of rank* ≥ 2.

Margulis space-times

- $\mathbf{Isom}^+(\mathsf{E})$ the group of Lorentzian isometries on the flat Lorentzian space E.
- A discrete affine group Γ acting properly on E is either solvable or is free of rank ≥ 2.
- Γ is a *proper affine free group of rank* ≥ 2.
- Assume for convenience $\mathcal{L}(\Gamma) \subset \mathsf{SO}(2,1)^o$. Γ is a *proper affine deformation*.

Margulis space-times

- $\mathbf{Isom}^+(\mathrm{E})$ the group of Lorentzian isometries on the flat Lorentzian space E.
- A discrete affine group Γ acting properly on E is either solvable or is free of rank ≥ 2.
- Γ is a *proper affine free group of rank* ≥ 2.
- Assume for convenience $\mathcal{L}(\Gamma) \subset \mathrm{SO}(2,1)^o$. Γ is a *proper affine deformation*.
- Assume $\mathcal{L}(\Gamma)$ is a free group of rank $\mathrm{g}, \mathrm{g} \geq 2$ in $\mathrm{SO}(2,1)^o$ acting freely and discretely on $\mathbb{H}^2$.

Real projective structures

- A *real projective structure* on a manifold is given by a maximal atlas of charts to $\mathbb{R}P^n, n \geq 1$, with transition maps in $\mathrm{PGL}(n+1, \mathbb{R})$.
- Suppose that Σ is a real projective surface with holonomy in the image of $\mathcal{L}(\Gamma)$ in $\mathrm{PSO}(2,1)$.
- A *parabolic annulus* in Σ is a properly embedded compact annulus with a parabolic holonomy.

Main Theorem

Theorem 2.1

Suppose that Γ *is a proper affine free group of rank* $\mathrm{g}, \mathrm{g} \geq 2$, *with parabolics and linear parts in* $\mathrm{SO}(2,1)^o$. *Then*

Main Theorem

Theorem 2.1

Suppose that Γ *is a proper affine free group of rank* $g, g \geq 2$*, with parabolics and linear parts in* $SO(2,1)^o$*. Then*

- E/Γ *is diffeomorphic to the interior of a compact handlebody of genus* g*.*

Main Theorem

Theorem 2.1

Suppose that Γ *is a proper affine free group of rank* $g, g \geq 2$*, with parabolics and linear parts in* $SO(2,1)^o$*. Then*

- E/Γ *is diffeomorphic to the interior of a compact handlebody of genus* g*.*
- *Moreover, it is the interior of a real projective 3-manifold M with a totally geodesic real projective surface as boundary.*

Main Theorem

Theorem 2.1

Suppose that Γ *is a proper affine free group of rank* $\mathrm{g}, \mathrm{g} \geq 2$*, with parabolics and linear parts in* $\mathrm{SO}(2,1)^o$*. Then*

- E/Γ *is diffeomorphic to the interior of a compact handlebody of genus* g*.*
- *Moreover, it is the interior of a real projective 3-manifold* M *with a totally geodesic real projective surface as boundary.*
- M *deformation retracts to a compact handlebody obtained by removing a union of finitely many solid-torus-end-neighborhoods.*

Remark 1

The tameness part is also claimed by Danciger, Kassel, and Guéritaud [5]. Also, the tameness without parabolics was also solved by Choi-Goldman and this group. Crooked plane conjecture for nonparabolic case was solved by this group also.

- We conjecture that the Margulis space-time with parabolics deforms immediately to one without parbolics. However, this requires result of Goldman-Labourie-Margulis-Minsky [8] which they have not written up.

- We conjecture that the Margulis space-time with parabolics deforms immediately to one without parbolics. However, this requires result of Goldman-Labourie-Margulis-Minsky [8] which they have not written up.
- The Crooked-plane conjecture is also claimed by DGK [5] and this should also imply the relative compactification.

- We conjecture that the Margulis space-time with parabolics deforms immediately to one without parbolics. However, this requires result of Goldman-Labourie-Margulis-Minsky [8] which they have not written up.
- The Crooked-plane conjecture is also claimed by DGK [5] and this should also imply the relative compactification.
- The main advantage of our approach is to see the 3-dimensional picture such as axes of transformations and globally hyperbolic subspaces bounded by Cauchy hypersurfaces. Also, relative compactification is easy to see.

- We conjecture that the Margulis space-time with parabolics deforms immediately to one without parbolics. However, this requires result of Goldman-Labourie-Margulis-Minsky [8] which they have not written up.
- The Crooked-plane conjecture is also claimed by DGK [5] and this should also imply the relative compactification.
- The main advantage of our approach is to see the 3-dimensional picture such as axes of transformations and globally hyperbolic subspaces bounded by Cauchy hypersurfaces. Also, relative compactification is easy to see.
- Also, these show that every flat complete Lorentz manifold of any dimension is tame. (Goldman-Labourie [6])

Real projective geometry of Margulis space-times

- Define
$$\mathbb{S}(V) := V \setminus \{0\}/\sim_+ \text{ where } \mathbf{x} \sim_+ \mathbf{y} \text{ iff } \mathbf{x} = s\mathbf{y} \text{ for } s \in \mathbb{R}_+.$$
There is a double cover $\mathbb{S}(V) \to \mathbb{P}(V)$ with the antipodal map $\mathcal{A} : \mathbb{S}(V) \to \mathbb{S}(V)$.
- $(\!(\mathbf{v})\!)$ denotes the equivalence class of $\mathbf{v}$.

Real projective geometry of Margulis space-times

- Define
$$\mathbb{S}(V) := V \setminus \{0\}/\sim_+ \text{ where } \mathbf{x} \sim_+ \mathbf{y} \text{ iff } \mathbf{x} = s\mathbf{y} \text{ for } s \in \mathbb{R}_+.$$
There is a double cover $\mathbb{S}(V) \rightarrow \mathbb{P}(V)$ with the antipodal map $\mathcal{A} : \mathbb{S}(V) \rightarrow \mathbb{S}(V)$.
- $(\!(\mathbf{v})\!)$ denotes the equivalence class of $\mathbf{v}$.
- $\mathrm{SL}_{\pm}(V)$ acts on $\mathbb{S}(V)$ effectively and transitively, and is $\mathbf{Aut}(\mathbb{S}(V))$.

Real projective geometry of Margulis space-times

- Define
$$\mathbb{S}(V) := V \setminus \{0\}/ \sim_+ \text{ where } \mathbf{x} \sim_+ \mathbf{y} \text{ iff } \mathbf{x} = s\mathbf{y} \text{ for } s \in \mathbb{R}_+.$$
There is a double cover $\mathbb{S}(V) \to \mathbb{P}(V)$ with the antipodal map $\mathcal{A} : \mathbb{S}(V) \to \mathbb{S}(V)$.
- $(\!(\mathbf{v})\!)$ denotes the equivalence class of $\mathbf{v}$.
- $\mathrm{SL}_{\pm}(V)$ acts on $\mathbb{S}(V)$ effectively and transitively, and is $\mathbf{Aut}(\mathbb{S}(V))$.
- E equals an open hemisphere in $\mathbb{S}^3 = \mathbb{S}(\mathbb{R}^4)$ by sending
$$(x_1, x_2, x_3) \text{ to } (\!(1, x_1, x_2, x_3)\!) \text{ for } x_1, x_2, x_3 \in \mathbb{R}.$$

Real projective geometry of Margulis space-times

- Define
$$\mathbb{S}(V) := V \setminus \{0\}/\sim_+ \text{ where } \mathbf{x} \sim_+ \mathbf{y} \text{ iff } \mathbf{x} = s\mathbf{y} \text{ for } s \in \mathbb{R}_+.$$
There is a double cover $\mathbb{S}(V) \to \mathbb{P}(V)$ with the antipodal map $\mathcal{A} : \mathbb{S}(V) \to \mathbb{S}(V)$.
- $(\!(\mathbf{v})\!)$ denotes the equivalence class of $\mathbf{v}$.
- $\mathrm{SL}_{\pm}(V)$ acts on $\mathbb{S}(V)$ effectively and transitively, and is $\mathbf{Aut}(\mathbb{S}(V))$.
- E equals an open hemisphere in $\mathbb{S}^3 = \mathbb{S}(\mathbb{R}^4)$ by sending
$$(x_1, x_2, x_3) \text{ to } (\!(1, x_1, x_2, x_3)\!) \text{ for } x_1, x_2, x_3 \in \mathbb{R}.$$
- $\partial \mathrm{E} = \partial \mathcal{H}$ is a great 2-sphere $\mathbb{S}$ given by $x_0 = 0$.

Real projective geometry of Margulis space-times

- Define
$$\mathbb{S}(V) := V \setminus \{0\}/\sim_+ \text{ where } \mathbf{x} \sim_+ \mathbf{y} \text{ iff } \mathbf{x} = s\mathbf{y} \text{ for } s \in \mathbb{R}_+.$$
There is a double cover $\mathbb{S}(V) \to \mathbb{P}(V)$ with the antipodal map $\mathcal{A} : \mathbb{S}(V) \to \mathbb{S}(V)$.
- $(\!(\mathbf{v})\!)$ denotes the equivalence class of $\mathbf{v}$.
- $\mathrm{SL}_\pm(V)$ acts on $\mathbb{S}(V)$ effectively and transitively, and is $\mathbf{Aut}(\mathbb{S}(V))$.
- E equals an open hemisphere in $\mathbb{S}^3 = \mathbb{S}(\mathbb{R}^4)$ by sending
$$(x_1, x_2, x_3) \text{ to } (\!(1, x_1, x_2, x_3)\!) \text{ for } x_1, x_2, x_3 \in \mathbb{R}.$$
- $\partial E = \partial\mathcal{H}$ is a great 2-sphere $\mathbb{S}$ given by $x_0 = 0$.
- $\mathbb{S} = \mathbb{S}_+ \cup \mathbb{S}_= \cup \mathbb{S}_0$.
- S_+ is the Klein model of the hyperbolic plane.

Hausdorff convergences

- $\mathbb{S}^3 = \mathbb{S}(\mathbb{R}^4)$ has Fubini-Study metric **d**.
- The *Hausdorff distance* between two compact sets A and B is

$$\mathbf{d}_H(A, B) = \inf\{\delta | \delta > 0, B \subset N_{\mathbf{d},\delta}(A), A \subset N_{\mathbf{d},\delta}(B)\}.$$

Hausdorff convergences

- $\mathbb{S}^3 = \mathbb{S}(\mathbb{R}^4)$ has Fubini-Study metric $\mathbf{d}$.
- The *Hausdorff distance* between two compact sets A and B is

$$\mathbf{d}_H(A, B) = \inf\{\delta | \delta > 0, B \subset N_{\mathbf{d},\delta}(A), A \subset N_{\mathbf{d},\delta}(B)\}.$$

Proposition 2.1 (see Benedetti-Petronio)

A sequence $\{A_i\}$ of compact sets converges to A in the Hausdorff topology if and only if

- *If there is a sequence $\{x_{i_j}\}$, $x_{i_j} \in A_{i_j}$, where $x_{i_j} \to x$ for $i_j \to \infty$, then $x \in A$.*
- *If $x \in A$, then there exists a sequence $\{x_i\}$, $x_i \in A_i$, such that $x_i \to x$.*

Linear parabolic action

- A linear endomorphism $N : V \to V$ is a *skew-adjoint endomorphism* of V if

$$\mathrm{B}(N\mathbf{x}, \mathbf{y}) = -\mathrm{B}(\mathbf{x}, N\mathbf{y}).$$

- We classify skew-adjoint linear parabolic transformations.

Linear parabolic action

- A linear endomorphism $N : V \to V$ is a *skew-adjoint endomorphism* of V if
$$\mathrm{B}(N\mathbf{x}, \mathbf{y}) = -\mathrm{B}(\mathbf{x}, N\mathbf{y}).$$
- We classify skew-adjoint linear parabolic transformations.

Corollary 3.1

Given a skew-adjoint endomorphism $N : V \to V$. Then there exists a coordinate system given by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfying

- - $\mathrm{B}(\mathbf{a}, \mathbf{b}) = 0 = \mathrm{B}(\mathbf{b}, \mathbf{c}), \mathrm{B}(\mathbf{a}, \mathbf{c}) = -1,$
  - $\mathbf{c} = N(\mathbf{b}), \mathbf{b} = N(\mathbf{a})$, *and*
  - $\mathbf{b}$ *is a unit spacelike vector,* $\mathbf{c} \in \mathrm{Ker} N$ *is casual null, and* $\mathbf{a}$ *is null.*

Linear parabolic action

- A linear endomorphism $N : V \to V$ is a *skew-adjoint endomorphism* of V if
$$\mathrm{B}(N\mathbf{x}, \mathbf{y}) = -\mathrm{B}(\mathbf{x}, N\mathbf{y}).$$
- We classify skew-adjoint linear parabolic transformations.

Corollary 3.1

Given a skew-adjoint endomorphism $N : V \to V$. Then there exists a coordinate system given by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ satisfying

-
    - $\mathrm{B}(\mathbf{a}, \mathbf{b}) = 0 = \mathrm{B}(\mathbf{b}, \mathbf{c}), \mathrm{B}(\mathbf{a}, \mathbf{c}) = -1$,
    - $\mathbf{c} = N(\mathbf{b}), \mathbf{b} = N(\mathbf{a})$, *and*
    - $\mathbf{b}$ *is a unit spacelike vector,* $\mathbf{c} \in \mathrm{Ker} N$ *is casual null, and* $\mathbf{a}$ *is null.*
- *The coordinate system is is canonical for a skew-symmetric nilpotent endomorphism N with respect to $\mathrm{B} : V \times V \to \mathbb{R}$.*

Proper affine parabolic action

- Let γ be an affine transformation with skew-adjoint parabolic linear part $\exp(N)$.

Proper affine parabolic action

- Let γ be an affine transformation with skew-adjoint parabolic linear part $\exp(N)$.
- Using the frame given as above and translating, γ lies in a one-parameter group

$$\Psi(t) := \exp t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2/2 & \mu t^3/6 \\ 0 & 1 & t & \mu t^2/2 \\ 0 & 0 & 1 & \mu t \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (3.1)$$

for $\mu \in \mathbb{R}$.

Proper affine parabolic action

- This one-parameter subgroup $\{\Psi(t), t \in \mathbb{R}\}$ leaves invariant the two polynomials

$$F_2(x,y,z) = z^2 - 2\mu y \text{ and } F_3(x,y,z) = z^3 - 3\mu yz + 3\mu^2 x, \tag{3.2}$$

and the diffeomorphism $F(x,y,z) := (F_3(x,y,z), F_2(x,y,z), z)$

$$F \circ \Psi(t) \circ F^{-1} : (x,y,z) \to (x,y,z+\mu t)\,. \tag{3.3}$$

Proper affine parabolic action

- This one-parameter subgroup $\{\Psi(t), t \in \mathbb{R}\}$ leaves invariant the two polynomials

$$F_2(x,y,z) = z^2 - 2\mu y \text{ and } F_3(x,y,z) = z^3 - 3\mu yz + 3\mu^2 x, \tag{3.2}$$

and the diffeomorphism $F(x,y,z) := (F_3(x,y,z), F_2(x,y,z), z)$

$$F \circ \Psi(t) \circ F^{-1} : (x,y,z) \to (x,y,z+\mu t)\,. \tag{3.3}$$

- All the orbits are twisted cubic curves.

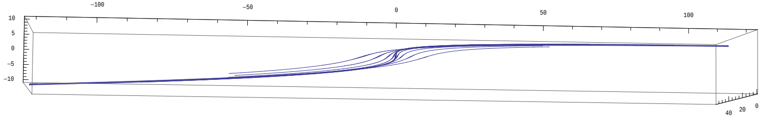


Figure: A number of orbits drawn horizontally.

Margulis invariants

- Let Γ be a proper affine deformation of a free group.
- For non-parabolic $\gamma \in \Gamma \setminus \{\mathrm{I}\}$, we define

Margulis invariants

- Let Γ be a proper affine deformation of a free group.
- For non-parabolic $\gamma \in \Gamma \setminus \{\mathrm{I}\}$, we define
  - $\mathbf{x}_+(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null directions with eigenvalue > 1,
  - $\mathbf{x}_-(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null direction with eigenvalue < 1, and

Margulis invariants

- Let Γ be a proper affine deformation of a free group.
- For non-parabolic $\gamma \in \Gamma \setminus \{\mathrm{I}\}$, we define
  - $\mathbf{x}_+(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null directions with eigenvalue > 1,
  - $\mathbf{x}_-(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null direction with eigenvalue < 1, and
  - $\mathbf{x}_0(\gamma)$ as the spacelike positive eigenvector of $\mathcal{L}(\gamma)$ of eigenvalue 1 given by

$$\mathbf{x}_0(\gamma) = \frac{\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)}{\|\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)\|}.$$

Margulis invariants

- Let Γ be a proper affine deformation of a free group.
- For non-parabolic $\gamma \in \Gamma \setminus \{\mathrm{I}\}$, we define
  - $\mathbf{x}_+(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null directions with eigenvalue > 1,
  - $\mathbf{x}_-(\gamma)$ as an eigenvector of $\mathcal{L}(\gamma)$ in the casual null direction with eigenvalue < 1, and
  - $\mathbf{x}_0(\gamma)$ as the spacelike positive eigenvector of $\mathcal{L}(\gamma)$ of eigenvalue 1 given by

$$\mathbf{x}_0(\gamma) = \frac{\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)}{\|\mathbf{x}_-(\gamma) \times \mathbf{x}_+(\gamma)\|}.$$

  - The Margulis invariant is given

$$\alpha(\gamma) = \mathsf{B}(\gamma(x) - x, \mathbf{x}_0(\gamma)), x \in \mathsf{E} \tag{3.4}$$

independent of the choice of x.

Charette-Drumm invariants $cd(\cdot)$

Definition 3.1

An eigenvector $\mathbf{v}$ of eigenvalue 1 of parabolic transformation g is *positive* relative to g if

- $\{\mathbf{v}, \mathbf{x}, \mathcal{L}(g)\mathbf{x}\}$ is positively oriented when
- $\mathbf{x}$ is any null or timelike vector which is not an eigenvector of g.

Definition 3.2

Charette-Drumm invariants $cd(\cdot)$

Definition 3.1

An eigenvector $\mathbf{v}$ of eigenvalue 1 of parabolic transformation g is *positive* relative to g if

- $\{\mathbf{v}, \mathbf{x}, \mathcal{L}(g)\mathbf{x}\}$ is positively oriented when
- $\mathbf{x}$ is any null or timelike vector which is not an eigenvector of g.

Definition 3.2

- Let $F(\mathcal{L}(g))$ be the eigensubspace of $\mathcal{L}(g)$ of eigenvalue 1.
- Define $\tilde{\alpha}(\gamma) : F(\mathcal{L}(\gamma)) \to \mathbb{R}$ by

$$\tilde{\alpha}(\gamma)(\cdot) = \mathsf{B}(\gamma(x) - x, \cdot), x \in \mathsf{E}.$$

Charette-Drumm invariants $cd(\cdot)$

Definition 3.1

An eigenvector $\mathbf{v}$ of eigenvalue 1 of parabolic transformation g is *positive* relative to g if

- $\{\mathbf{v}, \mathbf{x}, \mathcal{L}(g)\mathbf{x}\}$ is positively oriented when
- $\mathbf{x}$ is any null or timelike vector which is not an eigenvector of g.

Definition 3.2

- Let $F(\mathcal{L}(g))$ be the eigensubspace of $\mathcal{L}(g)$ of eigenvalue 1.
- Define $\tilde{\alpha}(\gamma) : F(\mathcal{L}(\gamma)) \to \mathbb{R}$ by

$$\tilde{\alpha}(\gamma)(\cdot) = \mathsf{B}(\gamma(x) - x, \cdot), x \in \mathsf{E}.$$

- $cd(\gamma) > 0$ if $\tilde{\alpha}(\gamma)$ is positive on positive eigenvectors in $F(\mathcal{L}(\gamma)) \setminus \{0\}$ ([1]).

Charette-Drumm invariants $cd(\cdot)$

Definition 3.1

An eigenvector $\mathbf{v}$ of eigenvalue 1 of parabolic transformation g is *positive* relative to g if

- $\{\mathbf{v}, \mathbf{x}, \mathcal{L}(g)\mathbf{x}\}$ is positively oriented when
- $\mathbf{x}$ is any null or timelike vector which is not an eigenvector of g.

Definition 3.2

- Let $F(\mathcal{L}(g))$ be the eigensubspace of $\mathcal{L}(g)$ of eigenvalue 1.
- Define $\tilde{\alpha}(\gamma) : F(\mathcal{L}(\gamma)) \to \mathbb{R}$ by

$$\tilde{\alpha}(\gamma)(\cdot) = \mathrm{B}(\gamma(x) - x, \cdot), x \in \mathrm{E}.$$

- $cd(\gamma) > 0$ if $\tilde{\alpha}(\gamma)$ is positive on positive eigenvectors in $F(\mathcal{L}(\gamma)) \setminus \{0\}$ ([1]).

Lemma 3.1

$\mu > 0$ if and only if $\gamma = \Phi_1$ has a positive Charette-Drumm invariant. Implying $\langle\gamma\rangle$ acts properly on E.

Constructing transversal foliations

- $\Psi(t) : \mathbf{E} \to \mathbf{E}$ is generated by a vector field

$$\phi := y\partial_x + z\partial_y + \mu\partial_z$$

with the square of the Lorentzian norm $\|\phi\|^2 = z^2 - 2\mu y$.

Constructing transversal foliations

- $\Psi(t) : \mathbf{E} \to \mathbf{E}$ is generated by a vector field

$$\phi := y\partial_x + z\partial_y + \mu\partial_z$$

with the square of the Lorentzian norm $\|\phi\|^2 = z^2 - 2\mu y$.

- Invariants of g^t are

$$F_2(x, y, z) = z^2 - 2\mu y \text{ and } F_3(x, y, z) = z^3 - 3\mu yz + 3\mu^2 x.$$

Constructing transversal foliations

- $\Psi(t) : \mathbf{E} \to \mathbf{E}$ is generated by a vector field

$$\phi := y\partial_x + z\partial_y + \mu\partial_z$$

  with the square of the Lorentzian norm $\|\phi\|^2 = z^2 - 2\mu y$.
- Invariants of g^t are

$$F_2(x, y, z) = z^2 - 2\mu y \text{ and } F_3(x, y, z) = z^3 - 3\mu yz + 3\mu^2 x.$$

- We define $\Psi(t, s) = g^t(l(s))$ so that

$$l(s) = (0, y_0, 0) + s(a, 0, c) = (sa, y_0, sc), \phi(l(s)) = (y_0, sc, \mu).$$

  ϕ is never parallel to $(a, 0, c)$ for $\frac{y_0}{\mu} < \frac{a}{c}$.

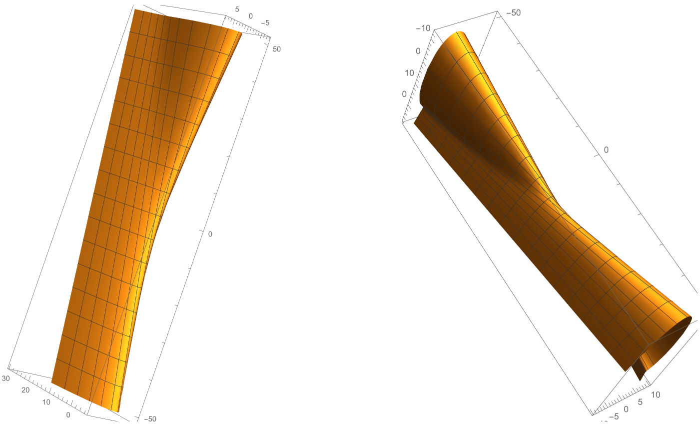


Figure: Two parabolic ruled surfaces. See [3].

Two transverse foliations.

- Assume $0 < \kappa_1 \leq \kappa_2 < \min\{1, \frac{3}{2\mu}\}$.
- Let $f : (0,1) \to \mathbb{R}$ be a strictly increasing analytic function satisfying

$$\kappa_1 \mu \frac{r}{\sqrt{1-r^2}} \leq f(r) \leq \kappa_2 \mu \frac{r}{\sqrt{1-r^2}}.$$

Two transverse foliations.

- Assume $0 < \kappa_1 \leq \kappa_2 < \min\{1, \frac{3}{2\mu}\}$.
- Let $f : (0,1) \to \mathbb{R}$ be a strictly increasing analytic function satisfying

$$\kappa_1 \mu \frac{r}{\sqrt{1-r^2}} \leq f(r) \leq \kappa_2 \mu \frac{r}{\sqrt{1-r^2}}.$$

- Let $\mathcal{H}_f$ be the space of compact segments u passing E with the following
  - ∂u in the horodisk $\mathcal{E} \subset \mathrm{Cl}(\mathbb{S}_+)$ containing $(\!(1,0,0)\!)$ in the boundary and in the antipodal set $\mathcal{E}_- \subset \mathrm{Cl}(\mathbb{S}_-)$,

Two transverse foliations.

- Assume $0 < \kappa_1 \leq \kappa_2 < \min\{1, \frac{3}{2\mu}\}$.
- Let $f : (0,1) \to \mathbb{R}$ be a strictly increasing analytic function satisfying

$$\kappa_1 \mu \frac{r}{\sqrt{1-r^2}} \leq f(r) \leq \kappa_2 \mu \frac{r}{\sqrt{1-r^2}}.$$

- Let $\mathcal{H}_f$ be the space of compact segments u passing E with the following
  - ∂u in the horodisk $\mathcal{E} \subset \mathrm{Cl}(\mathbb{S}_+)$ containing $(\!(1,0,0)\!)$ in the boundary and in the antipodal set $\mathcal{E}_- \subset \mathrm{Cl}(\mathbb{S}_-)$,
  - $u \cap \mathrm{E}$ is equivalent under g^t for some t to $l(s)$ given by $l_{f,r}(s) = (sa, y_f(r), sc), s \in \mathbb{R}$, where

$$y_f(r) := f(r), a = r, c = \sqrt{1-r^2}, r \in (0,1).$$

Two transverse foliations.

- Assume $0 < \kappa_1 \leq \kappa_2 < \min\{1, \frac{3}{2\mu}\}$.
- Let $f : (0,1) \to \mathbb{R}$ be a strictly increasing analytic function satisfying

$$\kappa_1 \mu \frac{r}{\sqrt{1-r^2}} \leq f(r) \leq \kappa_2 \mu \frac{r}{\sqrt{1-r^2}}.$$

- Let $\mathcal{H}_f$ be the space of compact segments u passing E with the following
  - ∂u in the horodisk $\mathcal{E} \subset \mathrm{Cl}(\mathbb{S}_+)$ containing $⟦1,0,0⟧$ in the boundary and in the antipodal set $\mathcal{E}_- \subset \mathrm{Cl}(\mathbb{S}_-)$,
  - $u \cap$ E is equivalent under g^t for some t to $l(s)$ given by $l_{f,r}(s) = (sa, y_f(r), sc), s \in \mathbb{R}$, where

$$y_f(r) := f(r), a = r, c = \sqrt{1-r^2}, r \in (0,1).$$

- For $r \in (0,1)$, let $S_{f,r}$ denote the *parabolic ruled surface* given by

$$\bigcup_{t,s\in\mathbb{R}} g^t(l_{f,r}(s)).$$

Remark 2

Define $D_{f,r_0,t}$ for $t \in \mathbb{R}$ denote the surface

$$\bigcup_{s\in\mathbb{R}, r\in[r_0,1)} g^t(l_{f,r}(s)).$$

Theorem 3.2

Let $r_0 \in (0,1)$. *Then the following hold*:

Remark 2

Define $D_{f,r_0,t}$ for $t \in \mathbb{R}$ denote the surface

$$\bigcup_{s\in\mathbb{R}, r\in[r_0,1)} g^t(l_{f,r}(s)).$$

Theorem 3.2

Let $r_0 \in (0,1)$. Then the following hold:

- *$S_{f,r}$ for $r \in [r_0, 1)$ are properly embedded leaves of a foliation $\tilde{S}_{f,r_0}$ of the region R_{f,r_0}, closed in* E*, bounded by S_{f,r_0} where g^t acts on.*

Remark 2

Define $D_{f,r_0,t}$ for $t \in \mathbb{R}$ denote the surface

$$\bigcup_{s\in\mathbb{R}, r\in[r_0,1)} g^t(l_{f,r}(s)).$$

Theorem 3.2

Let $r_0 \in (0,1)$. Then the following hold:

- *$S_{f,r}$ for $r \in [r_0, 1)$ are properly embedded leaves of a foliation $\tilde{\mathcal{S}}_{f,r_0}$ of the region R_{f,r_0}, closed in E, bounded by S_{f,r_0} where g^t acts on.*
- *$\{D_{f,r_0,t}, t \in \mathbb{R}\}$ is the set of properly embedded leaves of a foliation $\tilde{\mathcal{D}}_{f,r_0}$ of R_{f,r_0} by disks meeting $S_{f,r}$ for each r, $r_0 < r < 1$, transversally.*
  - $g^{t_0}(D_{f,r_0,t}) = D_{f,r_0,t+t_0}$.
  - $D_{f,r_0,t'} \cap D_{f,r_0,t} = \emptyset$ *for* $t, t', t \neq t'$.
  - $\mathrm{Cl}(D_{f,r_0,t}) \cap \mathbb{S}_+$ *is given as a geodesic ending at the parabolic fixed point of* g.

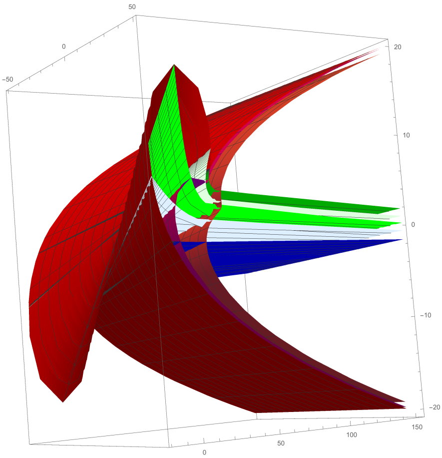


Figure: Three reddish leaves of foliation $\mathcal{S}_{f,r_0}$ and three bluish leaves of $\mathcal{D}_{f,r_0}$ where $f(r) = \frac{3}{4}\frac{r}{\sqrt{1-r^2}}$ and $\mu = 1$. See [4].

Tameness of $\mathsf{E}/\langle\gamma\rangle$

Definition 3.3

The quotient $R_{f,r_0}/\langle g\rangle$ is homeomorphic to a solid torus and is foliated by $\mathcal{S}_{f,r_0}$ induced by $\tilde{\mathcal{S}}_{f,r_0}$ and $\mathcal{D}_{f,r_0}$ induced by $\tilde{\mathcal{D}}_{f,r_0}$. The leaves of $\mathcal{S}_{f,r_0}$ are annuli of form $S_{f,r}/\langle g\rangle$, and the leaves of $\mathcal{D}_{f,r_0}$ are the embedded images of $D_{f,r_0,t}$ for $t\in\mathbb{R}$. The embedded image of $R_{f,r_0}/\langle g\rangle$ in E/Γ are foliated also.

Tameness of $\mathsf{E}/\langle\gamma\rangle$

Definition 3.3

The quotient $R_{f,r_0}/\langle g\rangle$ is homeomorphic to a solid torus and is foliated by $\mathcal{S}_{f,r_0}$ induced by $\tilde{\mathcal{S}}_{f,r_0}$ and $\mathcal{D}_{f,r_0}$ induced by $\tilde{\mathcal{D}}_{f,r_0}$. The leaves of $\mathcal{S}_{f,r_0}$ are annuli of form $S_{f,r}/\langle g\rangle$, and the leaves of $\mathcal{D}_{f,r_0}$ are the embedded images of $D_{f,r_0,t}$ for $t \in \mathbb{R}$. The embedded image of $R_{f,r_0}/\langle g\rangle$ in E/Γ are foliated also.

Theorem 3.4 (Parabolic Tameness)

Let γ be a parabolic affine transformation with a positive Charette-Drumm invariant. Then $\mathsf{E}/\langle\gamma\rangle$ is homeomorphic to a solid torus.

Tameness of $\mathsf{E}/\langle\gamma\rangle$

Definition 3.3

The quotient $R_{f,r_0}/\langle g\rangle$ is homeomorphic to a solid torus and is foliated by $\mathcal{S}_{f,r_0}$ induced by $\tilde{\mathcal{S}}_{f,r_0}$ and $\mathcal{D}_{f,r_0}$ induced by $\tilde{\mathcal{D}}_{f,r_0}$. The leaves of $\mathcal{S}_{f,r_0}$ are annuli of form $S_{f,r}/\langle g\rangle$, and the leaves of $\mathcal{D}_{f,r_0}$ are the embedded images of $D_{f,r_0,t}$ for $t \in \mathbb{R}$. The embedded image of $R_{f,r_0}/\langle g\rangle$ in E/Γ are foliated also.

Theorem 3.4 (Parabolic Tameness)

Let γ be a parabolic affine transformation with a positive Charette-Drumm invariant. Then $\mathsf{E}/\langle\gamma\rangle$ is homeomorphic to a solid torus.

Remark 3

We may use a γ-invariant foliation of E by crooked planes from the results of Charette-Kim [2]. We will give a topological proof later.

Anosov property of the geodesic flows

- Let Γ be as above with parabolics so that $M = \mathsf{E}/\Gamma$ is a Margulis space-time.
- Define $\mathbf{V}$ as a quotient budle of $\tilde{\mathbf{V}} := \mathsf{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ under the diagonal action

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), x \in \mathsf{U}\mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

Anosov property of the geodesic flows

- Let Γ be as above with parabolics so that $M = \mathsf{E}/\Gamma$ is a Margulis space-time.
- Define $\mathbf{V}$ as a quotient budle of $\tilde{\mathbf{V}} := \mathsf{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ under the diagonal action

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), x \in \mathsf{U}\mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

- The vector bundle $\mathbf{V}$ has a fiberwise Riemannian metric $\|\cdot\|_{\mathrm{fiber}}$ where Γ acts as isometries.
- Define $\widetilde{\mathscr{V}} := \mathbb{S}_+ \times \mathbb{R}^{2,1}$ and the bundle $\mathscr{V} := \widetilde{\mathscr{V}}/\Gamma$ with the action

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), x \in \mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

Anosov property of the geodesic flows

- Let Γ be as above with parabolics so that $M = \mathsf{E}/\Gamma$ is a Margulis space-time.
- Define $\mathbf{V}$ as a quotient budle of $\tilde{\mathbf{V}} := \mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ under the diagonal action

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), x \in \mathrm{U}\mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

- The vector bundle $\mathbf{V}$ has a fiberwise Riemannian metric $\|\cdot\|_{\mathrm{fiber}}$ where Γ acts as isometries.
- Define $\widetilde{\mathscr{V}} := \mathbb{S}_+ \times \mathbb{R}^{2,1}$ and the bundle $\mathscr{V} := \widetilde{\mathscr{V}}/\Gamma$ with the action

$$\gamma(x, \mathbf{v}) = (D\gamma(x), \mathcal{L}(\gamma)(\mathbf{v})), x \in \mathbb{S}_+, \mathbf{v} \in \mathbb{R}^{2,1}, \gamma \in \Gamma.$$

- Let $\Phi_t : \mathrm{U}\mathbb{S}_+ \to \mathrm{U}\mathbb{S}_+$ denote the geodesic flow on $\mathrm{U}\mathbb{S}_+$ defined by the hyperbolic metric.
- Let

$$D\Phi_t : \mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1} \to \mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$$

denote the flow acting trivially on the second factor and as the geodesic flow on $\mathrm{U}\mathbb{S}_+$.

Decomposition of $\mathbf{V}$

Given $(\langle\!\langle \mathbf{x} \rangle\!\rangle, \mathbf{u}) \in \mathrm{U}\mathbb{S}_+$,

- Define $l(\langle\!\langle \mathbf{x} \rangle\!\rangle, \mathbf{u}) \subset \mathbb{S}_+$ to be the oriented complete geodesic passing through $\langle\!\langle \mathbf{x} \rangle\!\rangle$ in the direction of $\mathbf{u}$, and
- Define $\mathbf{v}_{+,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})} = 1/\sqrt{2}\mathbf{j} + 1/\sqrt{2}\mathbf{k}$ and $\mathbf{v}_{-,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})} = -1/\sqrt{2}\mathbf{j} + 1/\sqrt{2}\mathbf{k}$ endpoints of the geodesic $l(\langle\!\langle \mathbf{k} \rangle\!\rangle, \mathbf{j}) \subset \mathbb{S}_+$.

Decomposition of $\mathbf{V}$

Given $(\langle\!\langle \mathbf{x} \rangle\!\rangle, \mathbf{u}) \in \mathrm{U}\mathbb{S}_+$,

- Define $l(\langle\!\langle \mathbf{x} \rangle\!\rangle, \mathbf{u}) \subset \mathbb{S}_+$ to be the oriented complete geodesic passing through $\langle\!\langle \mathbf{x} \rangle\!\rangle$ in the direction of $\mathbf{u}$, and
- Define $\mathbf{v}_{+,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})} = 1/\sqrt{2}\mathbf{j} + 1/\sqrt{2}\mathbf{k}$ and $\mathbf{v}_{-,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})} = -1/\sqrt{2}\mathbf{j} + 1/\sqrt{2}\mathbf{k}$ endpoints of the geodesic $l(\langle\!\langle \mathbf{k} \rangle\!\rangle, \mathbf{j}) \subset \mathbb{S}_+$.
- Define $\mathbf{v}_{+,(\langle\!\langle \mathbf{x} \rangle\!\rangle,\mathbf{u})}$ and $\mathbf{v}_{-,(\langle\!\langle \mathbf{x} \rangle\!\rangle,\mathbf{u})}$ respectively to be the images of $\mathbf{v}_{+,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})}$ and $\mathbf{v}_{-,(\langle\!\langle \mathbf{k} \rangle\!\rangle,\mathbf{j})}$ under $\mathcal{L}(g)$ if

$$\mathcal{L}(g)\,\langle\!\langle \mathbf{k} \rangle\!\rangle = \mathbf{x} \text{ and } Dg(\mathbf{j}) = \mathbf{u}.$$

We give as a basis

$$\left\{ \mathbf{v}_{+,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}, \mathbf{v}_{-,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}, \mathbf{v}_{0,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})} := \frac{\mathbf{v}_{-,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})} \times \mathbf{v}_{+,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}}{\left\| \mathbf{v}_{-,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})} \times \mathbf{v}_{+,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})} \right\|} \right\} \tag{4.1}$$

for the fiber over $\llparenthesis \mathbf{x} \rrparenthesis$ where $\times$ is the Lorentzian crossproduct.

- Let $\tilde{\mathbf{V}}_0$ be the 1-dimensional subbundle of $\mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ containing $\mathbf{v}_{0,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}$.
- Let $\tilde{\mathbf{V}}_+$ be the 1-dimensional subbundle of $\mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ containing $\mathbf{v}_{+,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}$.
- Let $\tilde{\mathbf{V}}_-$ be the 1-dimensional subbundle of $\mathrm{U}\mathbb{S}_+ \times \mathbb{R}^{2,1}$ containing $\mathbf{v}_{-,(\llparenthesis \mathbf{x} \rrparenthesis,\mathbf{u})}$.

Exponential stretching and contracting

Recall from Section 4.4 of [7] that the flow Φ_t acts on $\mathbf{V}$, and $\mathbf{V}$ splits into three Φ_t-invariant line bundles $\mathbf{V}_+$, $\mathbf{V}_-$ and $\mathbf{V}_0$, which are images of $\tilde{\mathbf{V}}_+$, $\tilde{\mathbf{V}}_-$ and $\tilde{\mathbf{V}}_0$.

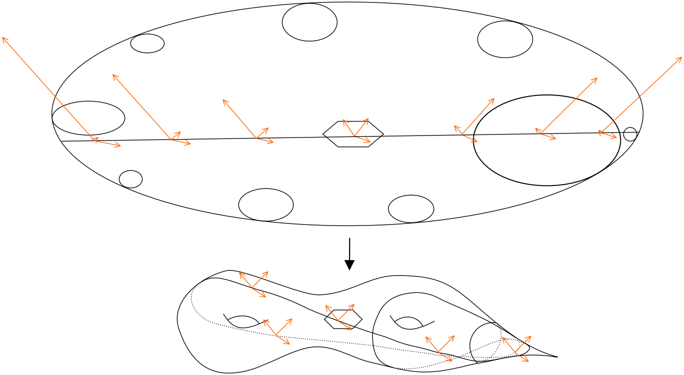
Exponential stretching and contracting

Recall from Section 4.4 of [7] that the flow Φ_t acts on $\mathbf{V}$, and $\mathbf{V}$ splits into three Φ_t-invariant line bundles $\mathbf{V}_+$, $\mathbf{V}_-$ and $\mathbf{V}_0$, which are images of $\tilde{\mathbf{V}}_+$, $\tilde{\mathbf{V}}_-$ and $\tilde{\mathbf{V}}_0$.

Our choice of $\|\cdot\|_{\mathrm{fiber}}$ shows that $D\Phi_t$ acts as uniform contraction in $\mathbf{V}_+$ as $t \to \infty, -\infty$, i.e.,

$$\begin{aligned} \|D\Phi_t(\mathbf{v}_+)\|_{\mathrm{fiber}} &\cong \exp(-t)\,\|\mathbf{v}_+\|_{\mathrm{fiber}} \text{ for } \mathbf{v}_+ \in \tilde{\mathbf{V}}_+, \\ \|D\Phi_t(\mathbf{v}_-)\|_{\mathrm{fiber}} &\cong \exp(t)\,\|\mathbf{v}_-\|_{\mathrm{fiber}} \text{ for } \mathbf{v}_- \in \tilde{\mathbf{V}}_-, \\ \|D\Phi_t(\mathbf{v}_0)\|_{\mathrm{fiber}} &\cong \|\mathbf{v}_0\|_{\mathrm{fiber}} \text{ for } \mathbf{v}_0 \in \tilde{\mathbf{V}}_0. \end{aligned} \tag{4.2}$$

Digram for bundles



The frames on $U\mathbb{S}_+$ and on US. The circles bound horodisks covering the cusp neighborhoods below. The compact set $\mathscr{K}$ is a some small compact set where the closed geodesics pass through.

de Rham isomorphism

- The $\mathscr{V}$-valued forms are differential forms with values in the fiber spaces of $\mathscr{V}$.
- The $\widetilde{\mathscr{V}}$-valued forms on $\mathbb{S}_+$ are simply the $\mathbb{R}^{2,1}$-valued forms on $\mathbb{S}_+$.

de Rham isomorphism

- The $\mathscr{V}$-valued forms are differential forms with values in the fiber spaces of $\mathscr{V}$.
- The $\widetilde{\mathscr{V}}$-valued forms on $\mathbb{S}_+$ are simply the $\mathbb{R}^{2,1}$-valued forms on $\mathbb{S}_+$.
- The group Γ acts by

$$\gamma^*(\mathbf{v} \otimes dx) = \mathcal{L}(\gamma)^{-1}(\mathbf{v}) \otimes d(x \circ \gamma), \gamma \in \Gamma. \tag{4.3}$$

- Write g as $g(x) = A_g x + \mathbf{b}_g$, $x \in \mathsf{E}$. Then $\mathbf{b} : \Gamma \to \mathbb{R}^{2,1}$ given by

$$g \mapsto \mathbf{b}_g \text{ for every } g$$

  is a cocycle representing an element of

$$H^1(\pi_1(\mathsf{S}), \mathbb{R}^{2,1}) = H^1(\mathsf{S}, \mathscr{V})$$

  using the de Rham isomorphism.

de Rham isomorphism

- The $\mathscr{V}$-valued forms are differential forms with values in the fiber spaces of $\mathscr{V}$.
- The $\widetilde{\mathscr{V}}$-valued forms on $\mathbb{S}_+$ are simply the $\mathbb{R}^{2,1}$-valued forms on $\mathbb{S}_+$.
- The group Γ acts by

$$\gamma^*(\mathbf{v} \otimes dx) = \mathcal{L}(\gamma)^{-1}(\mathbf{v}) \otimes d(x \circ \gamma), \gamma \in \Gamma. \tag{4.3}$$

- Write g as $g(x) = A_g x + \mathbf{b}_g$, $x \in \mathrm{E}$. Then $\mathbf{b} : \Gamma \to \mathbb{R}^{2,1}$ given by

$$g \mapsto \mathbf{b}_g \text{ for every } g$$

  is a cocycle representing an element of

$$H^1(\pi_1(\mathrm{S}), \mathbb{R}^{2,1}) = H^1(\mathrm{S}, \mathscr{V})$$

  using the de Rham isomorphism.
- Let η denote the smooth $\mathscr{V}$-valued 1-form on S representing the cocycle $\mathbf{b}$ in the de-Rham sense.

Estimating cocycle values $\mathbf{b}_g$

- We obtain

$$\mathbf{b}_g := \int_{[0,t_g]} D\Phi((x_g, \mathbf{u}_g), t)^{-1} \left(\tilde{\eta}\left(\frac{d\Phi((x_g, \mathbf{u}_g), t)}{dt}\right)\right) dt \tag{4.4}$$

where $\Phi((x_g, \mathbf{u}_g), [0, t_g])$ for $x_g \in \mathscr{K}$ and a unit vector $\mathbf{u}_g$ at x_g, covers a closed curve representing g.

Estimating cocycle values $\mathbf{b}_g$

- We obtain

$$\mathbf{b}_g := \int_{[0,t_g]} D\Phi((x_g, \mathbf{u}_g), t)^{-1} \left(\tilde{\eta}\left(\frac{d\Phi((x_g, \mathbf{u}_g), t)}{dt}\right)\right) dt \tag{4.4}$$

where $\Phi((x_g, \mathbf{u}_g), [0, t_g])$ for $x_g \in \mathscr{K}$ and a unit vector $\mathbf{u}_g$ at x_g, covers a closed curve representing g.

- Define

$$\tilde{\eta}_\omega(\langle\!\langle \mathbb{x} \rangle\!\rangle, \mathbf{u}) = \Pi_{\widetilde{\mathbf{V}}_\omega}(\tilde{\eta}(\langle\!\langle \mathbb{x} \rangle\!\rangle, \mathbf{u})), \tag{4.5}$$

where $\omega = +, -, 0$.

Estimating cocycle values $\mathbf{b}_g$

- We obtain

$$\mathbf{b}_g := \int_{[0,t_g]} D\Phi((x_g,\mathbf{u}_g),t)^{-1}\left(\tilde{\eta}\left(\frac{d\Phi((x_g,\mathbf{u}_g),t)}{dt}\right)\right)dt \tag{4.4}$$

where $\Phi((x_g,\mathbf{u}_g),[0,t_g])$ for $x_g \in \mathscr{K}$ and a unit vector $\mathbf{u}_g$ at x_g, covers a closed curve representing g.

- Define

$$\tilde{\eta}_\omega(\langle\!\langle \mathbb{x} \rangle\!\rangle,\mathbf{u}) = \Pi_{\widetilde{\mathbf{V}}_\omega}(\tilde{\eta}(\langle\!\langle \mathbb{x} \rangle\!\rangle,\mathbf{u})), \tag{4.5}$$

where $\omega = +,-,0$.

- We define invariants:

$$\mathbf{b}_{g,\omega} := \Pi_{\widetilde{\mathbf{V}}_\omega}(\mathbf{b}_g) = \int_{[0,t_g]} D\Phi((x_g,\mathbf{u}_g),t)^{-1}\left(\tilde{\eta}_\omega\left(\frac{d\Phi((x_g,\mathbf{u}_g),t)}{dt}\right)\right)dt, \tag{4.6}$$

- Let $\mathbf{H}_j \subset \mathbb{S}_+, j = 1, 2, \ldots,$ denote the horodisks Let p_j denote the parabolic fixed point corresponding to $\mathbf{H}_j$.

- Let $\mathbf{H}_j \subset \mathbb{S}_+, j = 1, 2, \ldots,$ denote the horodisks Let p_j denote the parabolic fixed point corresponding to $\mathbf{H}_j$.
- Each $\mathbf{H}_j$ has coordinates x_j, y_j from the upper half-space model where p_j becomes ∞, and $\mathbf{H}_j$ is given by $y_j > 1$.
- We may choose the 1-form η in the same cohomology class so that η', its lift to $\mathbb{S}_+$, is on any cusp neighborhood:

$$\mathbf{p}_j dx_j \text{ where } ⟪\mathbf{p}_j⟫ = p_j. \tag{4.7}$$

- Let $\mathbf{H}_j \subset \mathbb{S}_+, j = 1, 2, \ldots,$ denote the horodisks Let p_j denote the parabolic fixed point corresponding to $\mathbf{H}_j$.
- Each $\mathbf{H}_j$ has coordinates x_j, y_j from the upper half-space model where p_j becomes ∞, and $\mathbf{H}_j$ is given by $y_j > 1$.
- We may choose the 1-form η in the same cohomology class so that η', its lift to $\mathbb{S}_+$, is on any cusp neighborhood:

$$\mathbf{p}_j dx_j \text{ where } (\!(\mathbf{p}_j)\!) = p_j. \tag{4.7}$$

Theorem 4.1

Assume the positivity of Margulis and Charette-Drumm invariants, and $\mathcal{L}(\Gamma) \subset SO(2,1)^o$. For every sequence $\{g_i\}$ with $l(g_i) \to \infty$ of elements of $\Gamma_{\mathcal{K}}$, the following hold:

- $\|\mathbf{b}_{g_i}\|_E \to \infty$.
- $\{\|\mathbf{b}_{g_i -}\|_E\} < C$ *for a uniform constant* $C > 0$ *independent of* i.
- $\mathbf{d}((\!(\mathbf{b}_{g_i})\!), \mathrm{Cl}(\zeta_{a_{g_i}})) \to 0$.

Corollary 4.2

Let M be a Margulis space-time E/Γ with holonomy group Γ with parabolics. Let $K \subset \mathsf{E}$ be a compact subset. Let $y \in \mathbb{S}_+$, and let $\gamma_i \in \Gamma$ be a sequence such that $\gamma_i(y) \to y_\infty$ for $y_\infty \in \partial \mathbb{S}_+$. Then for every $\epsilon > 0$, there exists I_0 such that

$$\gamma_i(K) \subset N_{\mathbf{d},\epsilon}(\mathrm{Cl}(\zeta_{y_\infty})) \text{ for } i > I_0.$$

Equivalently, any sequence $\{\gamma_i(z_i) | z_i \in K\}$ accumulates only to $\mathrm{Cl}(\zeta_{y_\infty})$.

Exhaustions

Proposition 5.1 (Scott-Tucker)

Let E/Γ *be a Margulis space-time with parabolics. Then* E/Γ *has a sequence of handlebodies*

$$M_{(1)} \subset M_{(2)} \subset \cdots \subset M_{(i)} \subset M_{(i+1)} \subset \dots$$

so that $M_0 = \bigcup_{i=1}^{\infty} M_{(i)}$. *They have the following properties:*

- $\pi_1(M_{(1)}) \to \pi_1(M)$ *is an isomorphism.*
- *The inverse image* $\tilde{M}_{(i)}$ *of* $M_{(i)}$ *in* $\tilde{M}$ *is connected.*
- $\pi_1(M_{(i)}) \to \pi_1(M)$ *is surjective.*
- *for each compact subset* $K \subset \mathsf{E}/\Gamma$, *there exists an integer* I *so that for* $i > I$, $K \subset M_{(i)}$.

Boundedness in the parabolic sectors

Proposition 5.2

Let $\hat{R}_j$ denote the subdomain of the parabolic region R_j bounded by two crooked-boundary disks D_1 and D_2 whose closures contains $\mathrm{Cl}(\zeta_{p_j})$ for a parabolic fixed point p_j with the parabolic generator γ_j acting on R_j. Assume that $D_i \cap R_j, i = 1, 2$, is a ruled disk of the form of Theorem 3.2. Suppose that $D_1 \cap R_j$ and $\gamma_j^{\delta}(D_1) \cap R_j$ for $\delta = 1$ or -1 bounds a region in R_j containing $\hat{R}_j$.

Boundedness in the parabolic sectors

Proposition 5.2

Let $\hat{R}_j$ denote the subdomain of the parabolic region R_j bounded by two crooked-boundary disks D_1 and D_2 whose closures contains $\mathrm{Cl}(\zeta_{p_j})$ for a parabolic fixed point p_j with the parabolic generator γ_j acting on R_j. Assume that $D_i \cap R_j, i = 1, 2$, is a ruled disk of the form of Theorem 3.2. Suppose that $D_1 \cap R_j$ and $\gamma_j^{\delta}(D_1) \cap R_j$ for $\delta = 1$ or -1 bounds a region in R_j containing $\hat{R}_j$.

- *Then $\hat{R}_j \cap \tilde{M}_{(J)}$ is also compact for each j.*
- *Furthermore, we may assume that*

$$\tilde{M}_{(J)} \cap R_j = \emptyset \text{ for } j = 1, \ldots, c_0,$$

by choosing R_j sufficiently far away.

Boundedness in the parabolic sectors

Proposition 5.2

Let $\hat{R}_j$ denote the subdomain of the parabolic region R_j bounded by two crooked-boundary disks D_1 and D_2 whose closures contains $\mathrm{Cl}(\zeta_{p_j})$ for a parabolic fixed point p_j with the parabolic generator γ_j acting on R_j. Assume that $D_i \cap R_j, i = 1, 2$, is a ruled disk of the form of Theorem 3.2. Suppose that $D_1 \cap R_j$ and $\gamma_j^\delta(D_1) \cap R_j$ for $\delta = 1$ or -1 bounds a region in R_j containing $\hat{R}_j$.

- *Then $\hat{R}_j \cap \tilde{M}_{(J)}$ is also compact for each j.*
- *Furthermore, we may assume that*

$$\tilde{M}_{(J)} \cap R_j = \emptyset \text{ for } j = 1, \ldots, c_0,$$

by choosing R_j sufficiently far away.

Proof

We use exhaustions and Corollary 4.2

Choices of the candidate fundamental domain **F** bounded by almost crooked-disks $\mathcal{D}_j$

Now going to E/Γ with exhaustions $M_{(J)}$ as above.

Lemma 5.3

We can choose the mutually disjoint collection $\mathcal{D}_j \subset \mathsf{E}$ of properly embedded open disks and a tubular neighborhood $T_j \subset \mathrm{Cl}(\mathcal{D}_j)$ of $\partial\mathcal{D}_j$ for each j, $j = 1, \dots, 2\mathbf{g}$, that form a matching set $\{T_j | j = 1, \dots, 2\mathbf{g}\}$ for a collection $\mathcal{S}_0$ of generators of Γ. Finally, $\partial\mathcal{D}_j = d_j \cup \mathcal{A}(d_j) \cup \bigcup_{x \in \partial d_j} \mathrm{Cl}(\zeta_x)$ for a lift d_j of $\hat{d}_j$.

Choices of the candidate fundamental domain **F** bounded by almost crooked-disks $\mathcal{D}_j$

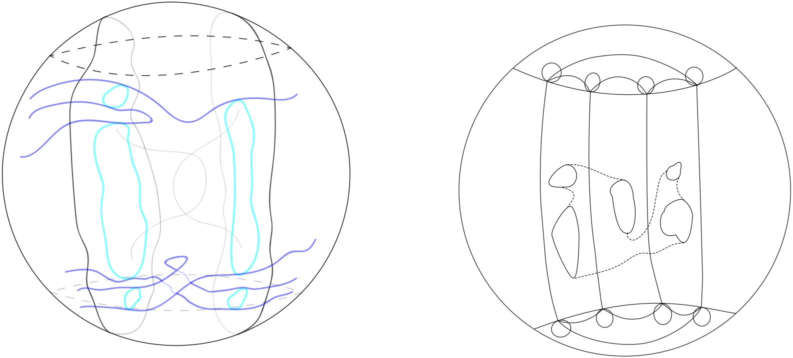
Now going to E/Γ with exhaustions $M_{(J)}$ as above.

Lemma 5.3

We can choose the mutually disjoint collection $\mathcal{D}_j \subset \mathsf{E}$ of properly embedded open disks and a tubular neighborhood $T_j \subset \mathrm{Cl}(\mathcal{D}_j)$ of $\partial\mathcal{D}_j$ for each j, $j = 1, \dots, 2\mathbf{g}$, that form a matching set $\{T_j | j = 1, \dots, 2\mathbf{g}\}$ for a collection $\mathcal{S}_0$ of generators of Γ. Finally, $\partial\mathcal{D}_j = d_j \cup \mathcal{A}(d_j) \cup \bigcup_{x\in\partial d_j} \mathrm{Cl}(\zeta_x)$ for a lift d_j of $\hat{d}_j$.

- Here, of course, the disk collection is not a matching set under $\mathcal{S}_0$.
- $\mathcal{D}_j$, $j = 1, 2, \dots, 2\mathbf{g}$, bound a region **F** closed in E with a compact closure in $\mathrm{Cl}(\mathsf{E})$, a finite-sided polytope in the topological sense.

Figures



(a) $\tilde{M}_{(J)}$ meeting with disks

(b) The fundamental domain bounded by disks

Tameness

Proposition 5.4 (Boundedness of $M_{(J)}$ in disks)

Let J be an arbitrary positive integer. For any crooked-boundary disk D, $D \cap \tilde{M}_{(J)}$ is compact, i.e., bounded, and has only finitely many components.

Proof.

Follows from Cor 4.2 and Prop. 5.2. □

Tameness

Proposition 5.4 (Boundedness of $M_{(J)}$ in disks)

Let J be an arbitrary positive integer. For any crooked-boundary disk D, $D \cap \tilde{M}_{(J)}$ is compact, i.e., bounded, and has only finitely many components.

Proof.

Follows from Cor 4.2 and Prop. 5.2. □

Definition 5.1

We modify T_j so that it is disjoint from the compact set in $\mathcal{D}_j$

$$\bigcup_{(k,l)\neq(j,j+\mathbf{g}) \mod 2\mathbf{g}} \mathcal{D}_j \cap \gamma_k(\mathcal{D}_l),$$

which we call an *unintended set*.

Tameness

Proposition 5.4 (Boundedness of $M_{(J)}$ in disks)

Let J be an arbitrary positive integer. For any crooked-boundary disk D, $D \cap \tilde{M}_{(J)}$ is compact, i.e., bounded, and has only finitely many components.

Proof.

Follows from Cor 4.2 and Prop. 5.2. □

Definition 5.1

We modify T_j so that it is disjoint from the compact set in $\mathcal{D}_j$

$$\bigcup_{(k,l)\neq(j,j+\mathbf{g}) \mod 2\mathbf{g}} \mathcal{D}_j \cap \gamma_k(\mathcal{D}_l),$$

which we call an *unintended set*.

- Now we consider K_0 be the set

$$\bigcup_{j=1}^{2\mathbf{g}} \bigcup_{(k,l)\neq(j,j+\mathbf{g}) \mod 2\mathbf{g}} (\mathcal{D}_j \cap \gamma_k(\mathcal{D}_l)).$$

which is a compact set by the finiteness. We also add to K_0 the following sets:

By Proposition 5.1, we choose $M_{(J)}$ in our exhaustion sequence of M so that

$$\tilde{M}_{(J)} \supset N_{\mathbf{d},\epsilon}(K_0) \tag{5.1}$$

for an ϵ-neighborhood, $\epsilon > 0$.

By Proposition 5.1, we choose $M_{(J)}$ in our exhaustion sequence of M so that

$$\tilde{M}_{(J)} \supset N_{\mathbf{d},\epsilon}(K_0) \tag{5.1}$$

for an ϵ-neighborhood, $\epsilon > 0$.

Lemma 5.5

$\tilde{M}_{(J)} \cap \mathcal{D}_i$ is a union of finitely many compact planar surfaces. Then $\bigcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i \cap \partial \tilde{M}_{(J)}$ maps to a union of embedded simple closed circles in $\partial M_{(J)}$ bounding immersed planar surfaces in $M_{(J)}$.

By Proposition 5.1, we choose $M_{(J)}$ in our exhaustion sequence of M so that

$$\tilde{M}_{(J)} \supset N_{\mathbf{d},\epsilon}(K_0) \tag{5.1}$$

for an ϵ-neighborhood, $\epsilon > 0$.

Lemma 5.5

$\tilde{M}_{(J)} \cap \mathcal{D}_i$ is a union of finitely many compact planar surfaces. Then $\bigcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i \cap \partial \tilde{M}_{(J)}$ maps to a union of embedded simple closed circles in $\partial M_{(J)}$ bounding immersed planar surfaces in $M_{(J)}$.

Proof.

This follows since they form the boundary of a fundamental region of $\partial \tilde{M}_{(J)}$. □

By Proposition 5.1, we choose $M_{(J)}$ in our exhaustion sequence of M so that

$$\tilde{M}_{(J)} \supset N_{\mathbf{d},\epsilon}(K_0) \tag{5.1}$$

for an ϵ-neighborhood, $\epsilon > 0$.

Lemma 5.5

$\tilde{M}_{(J)} \cap \mathcal{D}_i$ is a union of finitely many compact planar surfaces. Then $\bigcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i \cap \partial\tilde{M}_{(J)}$ maps to a union of embedded simple closed circles in $\partial M_{(J)}$ bounding immersed planar surfaces in $M_{(J)}$.

Proof.

This follows since they form the boundary of a fundamental region of $\partial\tilde{M}_{(J)}$. □

Proposition 5.6 (Outside Tameness)

Let M denote E/Γ where $\mathcal{L}(\Gamma) \subset \mathrm{SO}(2,1)^o$. Let $\mathbf{F}$ be the domain bounded by $\bigcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i$. Then $\mathbf{F} \setminus \tilde{M}_{(J)}$ is a fundamental domain of $M \setminus M_{(J)}$, and M is tame. Furthermore, $\bigsqcup_{i=1}^{2\mathbf{g}} \mathcal{D}_i \setminus \tilde{M}_{(J)}$ embeds to a disjoint union of properly embedded surfaces in M.

- By Dehn's lemma applied to $M_{(J)}$, each component of $\mathcal{D}_i \cap \partial\tilde{M}_{(J)}$ bounds a disk mapping to a mutually disjoint collection of embedded disks in $\tilde{M}_{(J)}$.
- We modify $\mathcal{D}_i$ by replacing each component of $\mathcal{D}_i \cap \tilde{M}_{(J)}$ with lifts of these disks.

- By Dehn's lemma applied to $M_{(J)}$, each component of $\mathcal{D}_i \cap \partial \tilde{M}_{(J)}$ bounds a disk mapping to a mutually disjoint collection of embedded disks in $M_{(J)}$.
- We modify $\mathcal{D}_i$ by replacing each component of $\mathcal{D}_i \cap \tilde{M}_{(J)}$ with lifts of these disks.
- We define $A_i := \bigcup_{x \in a_i} \zeta_x$, an open domain where ζ_x is the accordant semi-circle for x. We define

$$\tilde{\Sigma} := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i))$$

  for the antipodal map $\mathcal{A}$.
- $\Sigma := \tilde{\Sigma}/\Gamma$ is a real projective surface, i.e., the *ideal* $\mathbb{RP}^2$*-surface*.

- By Dehn's lemma applied to $M_{(J)}$, each component of $\mathcal{D}_i \cap \partial \tilde{M}_{(J)}$ bounds a disk mapping to a mutually disjoint collection of embedded disks in $M_{(J)}$.
- We modify $\mathcal{D}_i$ by replacing each component of $\mathcal{D}_i \cap \tilde{M}_{(J)}$ with lifts of these disks.
- We define $A_i := \bigcup_{x \in a_i} \zeta_x$, an open domain where ζ_x is the accordant semi-circle for x. We define

$$\tilde{\Sigma} := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i))$$

  for the antipodal map $\mathcal{A}$.
- $\Sigma := \tilde{\Sigma}/\Gamma$ is a real projective surface, i.e., the *ideal* $\mathbb{RP}^2$*-surface*.

Proposition 5.7

There exists a fundamental domain $\mathcal{R}$ *closed in* E *bounded by* $\mathcal{D}_j$, $j = 1, \dots, 2\mathbf{g}$. *Moreover,* $\mathrm{Cl}(\mathcal{R}) \cap (\mathsf{E} \cup \tilde{\Sigma})$ *is the fundamental domain of a manifold* $(\mathsf{E} \cup \tilde{\Sigma})/\Gamma$ *with boundary* Σ. *Here,* $\mathcal{R}$ *and* $\mathrm{Cl}(\mathcal{R})$ *are 3-cells, and* E/Γ *is homeomorphic to the interior of a handlebody of genus* $\mathbf{g}$.

Let $P = \bigcup_{\gamma \in \Gamma} \bigcup_{i=1,\dots,m_0} \gamma(\mathcal{P}_i)$, and let $\mathcal{P}_{\mathcal{R}} := (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{m_0}) \cap \mathcal{R}$.

Proposition 5.8

We can choose the sufficiently far away parabolic regions

$$\mathcal{P}_1, \dots, \mathcal{P}_{m_0}$$

meeting $\mathcal{R}$ *nicely so that they are disjoint in* E*. Then the following hold:*

Let $P = \bigcup_{\gamma\in\Gamma}\bigcup_{i=1,\ldots,m_0}\gamma(\mathcal{P}_i)$, and let $\mathcal{P}_{\mathcal{R}} := (\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{m_0}) \cap \mathcal{R}$.

Proposition 5.8

We can choose the sufficiently far away parabolic regions

$$\mathcal{P}_1, \ldots, \mathcal{P}_{m_0}$$

meeting $\mathcal{R}$ nicely so that they are disjoint in E*. Then the following hold:*

- $\gamma(\mathcal{P}_i) \cap \mathcal{R} \neq \emptyset$ *if and only if* $\gamma(\mathcal{P}_i)$ *meets* $\mathcal{R}$ *nicely, and* $\gamma(\mathcal{P}_i) = \mathcal{P}_j$ *for some* j.
- $\mathcal{R}$ *meets only* $\mathcal{P}_1, \ldots, \mathcal{P}_{m_0}$ *among all images* $\gamma(\mathcal{P}_r)$ *for* $\gamma \in \Gamma, r = 1, \ldots, m_0$.

Let $P = \bigcup_{\gamma\in\Gamma}\bigcup_{i=1,\dots,m_0}\gamma(\mathcal{P}_i)$, and let $\mathcal{P}_{\mathcal{R}} := (\mathcal{P}_1\cup\cdots\cup\mathcal{P}_{m_0})\cap\mathcal{R}$.

Proposition 5.8

We can choose the sufficiently far away parabolic regions

$$\mathcal{P}_1,\dots,\mathcal{P}_{m_0}$$

meeting $\mathcal{R}$ nicely so that they are disjoint in E. *Then the following hold:*

- $\gamma(\mathcal{P}_i)\cap\mathcal{R}\neq\emptyset$ *if and only if* $\gamma(\mathcal{P}_i)$ *meets* $\mathcal{R}$ *nicely, and* $\gamma(\mathcal{P}_i)=\mathcal{P}_j$ *for some* j.
- $\mathcal{R}$ *meets only* $\mathcal{P}_1,\dots,\mathcal{P}_{m_0}$ *among all images* $\gamma(\mathcal{P}_r)$ *for* $\gamma\in\Gamma, r=1,\dots,m_0$.
- *Moreover, for every pair* $\gamma,\eta\in\Gamma$,

$$\gamma(\mathcal{P}_j)\cap\eta(\mathcal{P}_k)=\emptyset \text{ or } \gamma(\mathcal{P}_j)=\eta(\mathcal{P}_k),\ j,k=1,\dots,m_0.$$

- First, we recall our bordifying surface:

$$\tilde{\Sigma}_0 := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i)).$$

- $\Sigma := \tilde{\Sigma}_0/\Gamma$ and $N := (\mathsf{E} \cup \tilde{\Sigma})/\Gamma$.

- First, we recall our bordifying surface:

$$\tilde{\Sigma}_0 := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i)).$$

- $\Sigma := \tilde{\Sigma}_0/\Gamma$ and $N := (\mathsf{E} \cup \tilde{\Sigma})/\Gamma$.
- We define P to be a union of mutually disjoint parabolic regions of form $\gamma(\mathcal{P}_i)$ for $\gamma \in \Gamma$, $i = 1, \dots, m_0$.
- We take the closure $\mathrm{Cl}(P)$ of P and take the relative interior P' in the closed hemisphere $\mathcal{H}$.
- Let $\partial_{\mathsf{E}} P'$ denote $\partial P \cap \mathsf{E}$. Then define $\tilde{N}' := \mathsf{E} \cup \tilde{\Sigma} \setminus P'$.

- First, we recall our bordifying surface:

$$\tilde{\Sigma}_0 := \mathbb{S}_+ \cup \mathbb{S}_- \cup \bigcup_{i \in \mathcal{I}} (A_i \cup a_i \cup \mathcal{A}(a_i)).$$

- $\Sigma := \tilde{\Sigma}_0/\Gamma$ and $N := (\mathsf{E} \cup \tilde{\Sigma})/\Gamma$.
- We define P to be a union of mutually disjoint parabolic regions of form $\gamma(\mathcal{P}_i)$ for $\gamma \in \Gamma$, $i = 1, \dots, m_0$.
- We take the closure $\mathrm{Cl}(P)$ of P and take the relative interior P' in the closed hemisphere $\mathcal{H}$.
- Let $\partial_{\mathsf{E}} P'$ denote $\partial P \cap \mathsf{E}$. Then define $\tilde{N}' := \mathsf{E} \cup \tilde{\Sigma} \setminus P'$.
- Γ acts properly discontinuously on $\tilde{N}'$. Thus, $N' := \tilde{N}'/\Gamma$ is a manifold.
- The manifold boundary $\partial N'$ of N' is

$$((\tilde{\Sigma} \setminus P') \cup \partial_{\mathsf{E}} P')/\Gamma.$$

  Define $P'' = P'/\Gamma$.

- Also, $\partial_N(P'') := (\partial_{\mathsf{E}} P')/\Gamma$ is a union of a finite number of disjoint annuli. $\partial N'$ is homeomorphic to $(\Sigma \setminus P'') \cup \partial_N P''$.
- Recall that the union of facial-disks $\mathcal{D}_i$, $i = 1, \ldots, 2\mathbf{g}$, bounds the fundamental domain $\mathcal{R}$ in $\mathcal{H}$.

- Also, $\partial_N(P'') := (\partial_{\mathsf{E}} P')/\Gamma$ is a union of a finite number of disjoint annuli. $\partial N'$ is homeomorphic to $(\Sigma \setminus P'') \cup \partial_N P''$.
- Recall that the union of facial-disks $\mathcal{D}_i$, $i = 1, \dots, 2\mathbf{g}$, bounds the fundamental domain $\mathcal{R}$ in $\mathcal{H}$.
- $$\bigcup_{i=1}^{2\mathbf{g}} \mathrm{Cl}(\mathcal{D}_i) \cap (\mathsf{E} \cup \tilde{\Sigma} \setminus P')$$
  bounds a fundamental domain
  $$\mathrm{Cl}(\mathcal{R}) \cap (\mathsf{E} \cup \tilde{\Sigma} \setminus P').$$

- Also, $\partial_N(P'') := (\partial_{\mathsf{E}} P')/\Gamma$ is a union of a finite number of disjoint annuli. $\partial N'$ is homeomorphic to $(\Sigma \setminus P'') \cup \partial_N P''$.
- Recall that the union of facial-disks $\mathcal{D}_i$, $i = 1, \ldots, 2\mathbf{g}$, bounds the fundamental domain $\mathcal{R}$ in $\mathcal{H}$.
- $$\bigcup_{i=1}^{2\mathbf{g}} \mathrm{Cl}(\mathcal{D}_i) \cap (\mathsf{E} \cup \tilde{\Sigma} \setminus P')$$
  bounds a fundamental domain
  $$\mathrm{Cl}(\mathcal{R}) \cap (\mathsf{E} \cup \tilde{\Sigma} \setminus P').$$
- $N' := (\mathsf{E} \cup \tilde{\Sigma} \setminus P')/\Gamma$ is compact and is homeomorphic to a handlebody of genus g by Theorem 5.2 of Hempel [9].
- N deformation retracts to N' as above since ϕ does not act on any component of P'.

Bibliography I

Virginie Charette and Todd A. Drumm.
The Margulis invariant for parabolic transformations.
Proc. Amer. Math. Soc., 133(8):2439–2447, 2005.

Virginie Charette and Youngju Kim.
Foliations of Minkowski $2+1$ spacetime by crooked planes.
Internat. J. Math., 25(9):1450088, 25, 2014.

Suhyoung Choi.
convergenceii.nb.
mathematica file at mathsci.kaist.ac.kr/~schoi/research.html.

Suhyoung Choi.
Foliationsv2.nb.
mathematica file at mathsci.kaist.ac.kr/~schoi/research.html.

Jeffrey Danciger, François Guéritaud, and Fanny Kassel.
Margulis spacetimes with parabolic elements.
in preparation.

William M. Goldman and François Labourie.
Geodesics in Margulis spacetimes.
Ergodic Theory Dynam. Systems, 32(2):643–651, 2012.

Bibliography II

William M. Goldman, François Labourie, and Gregory Margulis.
Proper affine actions and geodesic flows of hyperbolic surfaces.
Ann. of Math. (2), 170(3):1051–1083, 2009.

William M. Goldman, François Labourie, Gregory Margulis, and Yair Minsky.
Complete flat Lorentz 3-manifolds and laminations on hyperbolic surfaces.
in preparation, 2012.

John Hempel.
3-manifolds.
AMS Chelsea Publishing, Providence, RI, 2004.
Reprint of the 1976 original.