Closed affine manifolds with partially hyperbolic linear holonomy (preliminary)

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Abstract

- We give some introduction to the field of complete affine n-manifolds.
- We will try to show that closed manifolds of negative curvature do not admit complete special affine structures whose linear parts are partially hyperbolic in the dynamical sense.

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- We give some introduction to the field of complete affine n-manifolds.
- We will try to show that closed manifolds of negative curvature do not admit complete special affine structures whose linear parts are partially hyperbolic in the dynamical sense.
- We can drop the negative curvature condition. We present our attempt here.
- Partially a joint work with Kapovich.

Geometric structures

First, we give some introduction.

- G a Lie group acting transitively faithfully on a space X.
- Let M be a (probably closed) manifold. A (G, X)-structure is a maximal atlas of charts so that transition maps are in G.

Geometric structures

First, we give some introduction.

- G a Lie group acting transitively faithfully on a space X.
- Let M be a (probably closed) manifold. A (G, X)-structure is a maximal atlas of charts so that transition maps are in G.
- This is equivalent to M having a pair $(\mathbf{dev}, h)$
  - There is a homomophism $h : \pi_1(M) \to G$ called a *holonomy homomorphism*.
  - There is an immersion $\mathbf{dev} : \tilde{M} \to X$, called a *developing map*, so that

$$\mathbf{dev} \circ \gamma = h(\gamma) \circ \mathbf{dev} \text{ for each deck transformation } \gamma \in \pi_1(M).$$

Bundles and sections (See Goldman [5])

- Construct $X_h = \tilde{M} \times X/\pi_1(M)$ where $g(x, y) = (g(x), h(g)(x))$. This is a fiber bundle over M with fibers X.
- X_h is a bundle over M with a flat connection induced from the product structure.

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- X_h is a bundle over M with a flat connection induced from the product structure.
- There is a *developing* section $s : M \to X_h$ given by $\tilde{M} \ni x \mapsto (x, \mathbf{dev}(x)) \in \tilde{M} \times X$. The section is transverse to the flat connection.
- Conversely, a transverse section $s : M \to X_h$ gives us a (G, X)-structure.

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Complete (G, X)-structures

- Suppose that **dev** $: \tilde{M} \to X$ is a diffeomorphism. Then M is *complete*.
- We have a diffeomorphism $M \to X/h(\pi_1(M))$, and $h(\pi_1(M))$ acts properly discontinously and freely on X.
- Complete (G, X)-structures on M are classified by the conjugacy classes of $\pi_1(M) \to G$.

Affine manifolds

- Let $\mathbb{A}^n$ be a complete affine space. Let $\mathbf{Aff}(\mathbb{A}^n)$ denote the group of affine transformations of $\mathbb{A}^n$ whose elements are of form:

$$x \mapsto Ax + \mathbf{v}$$

  for a vector $\mathbf{v} \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n, \mathbb{R})$.
- Let $\mathcal{L} : \mathbf{Aff}(\mathbb{A}^n) \to \mathrm{GL}(n, \mathbb{R})$ denote map sending elements of $\mathbf{Aff}(\mathbb{A}^n)$ to its linear part in $\mathrm{GL}(n, \mathbb{R})$.

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- Example: $\mathbb{Z}^n$ acting on $\mathbb{A}^n$ as a translation group in lattice directions. The quotients are homeomorphic to T^n.
- Any Euclidean manifold is an affine manifold is finitely covered by $T^i \times \mathbb{R}^{n-i}$ for some i. (Bieberbach)

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    - A *complete affine n-manifold* is an n-manifold M of form $\mathbb{A}^n/\Gamma$.
    - Note that completeness and compactness of M have no relation (The Hopf-Rinow lemma does not hold here)

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Affine Solv 3-manifold

$$T_1 := (x, y, z) \mapsto (x+1, y, z),$$
$$T_2 := (x, y, z) \mapsto (x, y+1, z),$$
$$T_3 := (x, y, z) \mapsto (A(x, y), z+1) \quad (1)$$

where A is an special integral 2×2-matrix, e.g., $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\mathbb{A}^n / \langle T_1, T_2, T_3 \rangle$ is a mapping torus of Anosov diffeomorphism $T^2 \to T^2$.

Noncompact examples

Existence of actions

A properly discontinuous action on $\mathbb{A}^n$ of an affine group gives us examples of complete affine n-manifolds.

- Margulis, Drumm found first examples of free groups of rank ≥ 1 acting freely and properly on $\mathbb{A}^n$. These gives examples of complete affine 3-manifolds homeomorphic to handlebodies.

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- Danciger,Kassel, Gueritaud for large n for many Coxeter groups of hyperbolic types. They produce many complete affine manifolds, which are probably tame.
- By their work, there is a free action of $\mathbb{R}^n$ by many general manifold groups of negative curvature. Here n depends on the group.

Nonexistence of proper actions

- Danciger and Zhang [3] showed that when M is a surface, there is no properly discontinuous action on $\mathbb{R}^n$ by an affine representation with linear part in a Hitchin component.
- Ghosh [4] obtained some generalization to hyperbolic groups with affine representations with Anosov linear part.
- Tsouvalas: some cases must virtually be free or be a surface group.

However, these work do not have our dimension conditions.

Auslander Conjecture

Closed complete affine n-manifolds have virtually solvable fundamental groups.

- This is proved for $n = 2$ by Nagano-Yagi, $n = 3$ by Fried-Goldman, 1983, and for $n \leq 6$ for Abels-Margulis-Soifer.

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- Linear holonomy in $SO(p, q)$ implies the virtually solvable fundamental group. This is shown by Goldman-Kamishima 84 for $p = n - 1, q = 1$ and Abels-Margulis-Soifer some other cases including some cases of $p = n, q = n - 1$.

Partial hyperbolicity

- Denote by $\tilde{M}$ the universal cover of M with the covering map p_M with the deck transformation group $\pi_1(M)$.
- Let $\pi_M : \mathbf{U}M \to M$ denote the fibration and $\tilde{\pi}_M : \mathbf{U}\tilde{M} \to \tilde{M}$ the induced fibration.
- There is a covering $\mathbf{U}p_M : \mathbf{U}\tilde{M} \to \mathbf{U}M$ from the unit tangent bundle $\mathbf{U}\tilde{M}$ of $\tilde{M}$. The deck transformation group of $\mathbf{U}p_M$ is $\pi_1(M)$.

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- (Affine bundle): For an affine representation $\rho' : \pi_1(M) \to \mathbf{Aff}(\mathbb{A}^n)$, define $\mathbb{A}^n_{\rho'} := (\mathbf{U}\tilde{M} \times \mathbb{A}^n)/\pi_1(M)$ with the diagonal action.
- (Vector bundle): We define $\mathbb{R}^n_\rho := (\mathbf{U}\tilde{M} \times \mathbb{R}^n)/\pi_1(M)$ for $\rho = \mathcal{L} \circ \rho'$.

Flows lifted to the bundle

- Let $\hat{\phi}_t : \mathbf{U}M \to \mathbf{U}M$ denote the geodesic flow, and $\phi_t : \mathbf{U}\tilde{M} \to \mathbf{U}\tilde{M}$ denote the flow lifted from $\hat{\phi}_t$.
- There exists a flow $\Phi_t, t \in \mathbb{R}$, on $\mathbb{A}^n_{\rho'}$, acting as the geodesic flow ϕ_t on $\mathbf{U}M$ and acting trivially on $\mathbb{A}^n$ lifted.
- Also, there is a flow $D\Phi_t, t \in \mathbb{R}$, on $\mathbb{R}^n_\rho$ taking the linear part of Φ_t fiberwise acting as the geodesic flow on $\mathbf{U}M$ and acting trivially on $\mathbb{R}^n$ lifted.

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We have fiber-wise norm $\|\cdot\|_{\mathbb{A}^n_{\rho'}}$ on $\mathbb{A}^n_{\rho'}$, and a norm $\|\cdot\|_{\mathbb{R}^n_\rho}$ on $\mathbb{R}^n_\rho$ using partition of unity.

Partial hyperbolicity in the bundle sense.

- A representation $\rho : \pi_1(M) \to \mathrm{GL}(n, \mathbb{R})$ is *partially hyperbolic in a bundle sense* if the following hold:
  - (i) There exist C^0-subbundles $\mathbb{V}_+, \mathbb{V}_0$, and $\mathbb{V}_-$ in $\mathbb{R}^n_\rho$ invariant under the flow $D\Phi_t$.
  - (ii) $\mathbb{V}_+, \mathbb{V}_0$ and $\mathbb{V}_-$ are independent and their bundle sum equals $\mathbb{V}$.

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  (ii) $\mathbb{V}_+, \mathbb{V}_0$ and $\mathbb{V}_-$ are independent and their bundle sum equals $\mathbb{V}$.
  (iii) For any fiber-wise metric on $\mathbb{R}^n_\rho$ over $\mathbf{U}M$, the lifted action of $D\Phi_t$ on $\mathbb{V}_+$ (resp. $\mathbb{V}_-$) is dilating (resp. contracting): i.e., there are coefficients $A > 0$, $a > 0$, $A' > 0$:
    1. $||D\Phi_{-t}(\mathbf{v})||_{\mathbb{R}^n_\rho, \Phi_{-t}(m)} \leq A \exp(-at) \, ||\mathbf{v}||_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_+(m)$ as $t \to \infty$.
    2. $||D\Phi_t(\mathbf{v})||_{\mathbb{R}^n_\rho, \Phi_t(m)} \leq A \exp(-at) \, ||\mathbf{v}||_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_-(m))$ as $t \to \infty$.

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    2. $\|D\Phi_t(\mathbf{v})\|_{\mathbb{R}^n_\rho, \Phi_t(m)} \leq A \exp(-at) \|\mathbf{v}\|_{\mathbb{R}^n_\rho, m}$ for $\mathbf{v} \in \mathbb{V}_-(m))$ as $t \to \infty$.
    3. (A dominance property)

$$\frac{\|D\Phi_t(\mathbf{w})\|_{\mathbb{R}^n_\rho, \phi_t(m)}}{\|D\Phi_t(\mathbf{v})\|_{\mathbb{R}^n_\rho, \phi_t(m)}} \leq A' \exp(-a' t) \frac{\|\mathbf{w}\|_{\mathbb{R}^n_\rho, m}}{\|\mathbf{v}\|_{\mathbb{R}^n_\rho, m}} \begin{cases} \text{for } \mathbf{v} \in \mathbb{V}_+(m), \mathbf{w} \in \mathbb{V}_0(m) \text{ as } t \to \infty, \\ \text{or for } \mathbf{v} \in \mathbb{V}_0(m), \mathbf{w} \in \mathbb{V}_-(m) \text{ as } t \to \infty. \end{cases} \quad (2)$$

- Here $\dim \mathbb{V}_+$ is a *partial hyperbolicity index* of ρ.
- We assume that $\dim \mathbb{V}_+ = \dim \mathbb{V}_- \geq 1$. Also, $\mathbb{V}_0$ is said to be the *neutral subbundle* of $\mathbb{V}$. Often we will be in cases $\dim \mathbb{V}_0 > 0$.
- A related dynamical system is “partially hyperbolic system" as in Bonatti, Diaz, Viana [1] or Crovisier and Potrie [2]. (Related to Bochi-Sambarino and see Definition 1.5 of [2].)

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Theorem 1 (Negative curvature case)

Let M be a closed complete special affine n-manifold. Suppose that M admits a negatively curved Riemannian metric. Then the linear part of a holonomy homomorphism ρ is not a partially hyperbolic representation in a bundle sense.

Consequences

Question

We think that P-Anosov condition implies partially hyperbolic linear holonomy for every parabolic subgroup P of $\mathrm{SL}(n, \mathbb{R})$ in most situations. Consequently, every complete special affine closed manifold is not P-Anosov. (Maybe with a few exceptions for reducible $\mathcal{L} \circ \rho'$)

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Corollary 1 (Special Lie groups)

Let M be a closed complete special affine n-manifold with a fundamental group $\pi_1(M)$ with linear holonomy in $G = \mathrm{SO}(k, n-k)$ for $0 \leq k \leq n$ or $\mathrm{SP}(m, \mathbb{R})$ for $n = 2m$. Suppose that M admits a negatively curved Riemannian metric. Then the linear part of the holonomy homomorphism ρ is not P-Anosov for any parabolic group P of $\mathrm{SL}(n, \mathbb{R})$.

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Proof.

When ρ has images in the specified groups in the premises, the singular values are invariant under inverses. □

Developing sections

- We begin the proof of Theorem 1 for $M = \mathbb{A}^n/\Gamma$ for $\Gamma = \rho'(\pi_1(M))$.
- Let d_M denote the negatively curved Riemannian metric on M and on $\tilde{M}$.
- There is a projection $\tilde{\Pi}_{\mathbb{A}^n} : \mathbf{U}\tilde{M} \times \mathbb{A}^n \to \mathbb{A}^n$ inducing a bundle map

$$\Pi_{\mathbb{A}^n} : \mathbb{A}^n_{\rho'} := (\mathbf{U}\tilde{M} \times \mathbb{A}^n)/\pi_1(M) \to \mathbb{A}^n/\Gamma$$

and $\tilde{\pi}_{\mathbf{U}M} : \mathbf{U}\tilde{M} \times \mathbb{A}^n \to \mathbf{U}\tilde{M}$ inducing a bundle map

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- $d_{\mathbb{A}^n/\Gamma}$ denote one induced from d_M and $d_{\mathbb{A}^n}$ denote the lifted on on $\tilde{M}$.
- We define a section $\tilde{s} : \mathbf{U}\tilde{M} \to \mathbf{U}\tilde{M} \times \mathbb{A}^n$ where

$$\tilde{s}((x, \vec{v})) = ((x, \vec{v}), \mathbf{dev}(x)), x \in \tilde{M}. \tag{3}$$

- $\tilde{s}$ induces a section $s : \mathbf{U}M \to \mathbb{A}^n_{\rho'}$, called the *developing section*.

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- $\tilde{s}$ induces a section $s : \mathbf{U}M \to \mathbb{A}^n_{\rho'}$, called the *developing section*.
- Since $M = \mathbb{A}^n/\Gamma$ has a complete affine structure, **dev** induces the map

$$\mathcal{I} := \Pi_{\mathbb{A}^n} \circ s : \mathbf{U}M \to \mathbb{A}^n/\Gamma.$$

Neutralizing the sections

Proposition 2

There is a section $s_\infty : M \to \mathbb{A}^n_{\rho'}$*, homotopic to the developing section s in the* C^0*-topology with the following conditions:*

- $\nabla_\phi s_\infty$ *is in* $V_0(x)$ *for each* $x \in \mathbf{U}M$.
- $\mathcal{I}_\infty := \Pi_{\mathbb{A}^n} \circ s_\infty$ *is onto.*
- $d_{\mathbb{A}^n/\Gamma}(\tilde{s}(x), \tilde{s}_\infty(x))$ *and* $d_{\mathbb{A}^n/\Gamma}(\tilde{\mathcal{I}}(x), \tilde{\mathcal{I}}_\infty(x))$ *are uniformly bounded for* $x \in \mathbf{U}\tilde{M}$.

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Proof.

We project to flat connections $\nabla^+, \nabla^-, \nabla^0$ respectively on $\mathbb{V}_+, \mathbb{V}_0, \mathbb{V}_-$ respectively.
We define $s_\infty := s + \int_0^\infty (D\Phi_t)_*(\nabla_\phi^- s)dt - \int_0^\infty (D\Phi_{-t})_*(\nabla_\phi^+ s)dt$. Then it is homotopic to s since we can replace ∞ by T, $T > 0$ and let $T \to \infty$. (See Section 8 of Goldman-Labourie-Margulis [6].) Since M is compact and the norms of the integrand decreases exponentially, the integral is uniformly bounded above.

□

Corollary 2

$\tilde{\mathcal{I}_\infty} := \tilde{\Pi}_{\mathbb{A}^n} \circ \tilde{s}_\infty$ *restricted to each oriented geodesic* $\vec{l}$ *on* $\mathbf{U}\tilde{M}$ *lies on a neutral affine subspace parallel to* $V_0(\vec{l})$.

- Let $l_y := \{\phi_t(y) | t \geq 0\}$ for $y \in K$.
- The image $\tilde{\mathcal{I}_\infty}(l_y)$ is in a neutral affine subspace denoted it by A^0_y or $A^0_{l_y}$.
- We choose l_y so that an infinite-order deck-transformation γ acts on the axis containing l_y.

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- We choose l_y so that an infinite-order deck-transformation γ acts on the axis containing l_y.
- $$\tilde{s}_\infty \circ \gamma = \rho'(\gamma) \circ \tilde{s}_\infty, \gamma \in \pi_1(M) \text{ implies} \tag{4}$$
  $$\rho'(\gamma)(A^0_y) = A^0_{\gamma(y)} = \rho'(\gamma)(A^0_{l_y}) = A^0_{\gamma(l_y)}. \tag{5}$$

  In particular, γ acts on the axis containing l_y and on A^o_y.

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  In particular, γ acts on the axis containing l_y and on A^o_y.
- Finally since s_∞ is continuous, $x \mapsto A^0_x$ is a continuous function. Hence,

$$A^0_{z_i} \to A^0_z \text{ if } z_i \to z \in \mathbf{U}\tilde{M}. \tag{6}$$

- Denote by $\mathbb{V}_{\pm}(y)$ be the vector subspace parallel to the lift of $\mathbb{V}_{\pm}$ at y. The C^0-decomposition property also implies

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Proposition 3

$\tilde{\mathcal{I}_\infty}(\mathcal{R}_p)$ *equals* $\mathbb{A}^n$.

Definition 1

A_p^{0-}: the affine subspace containing A_p^0 and all points in directions of $\mathbb{V}^-(p)$ from points of A_p^0.

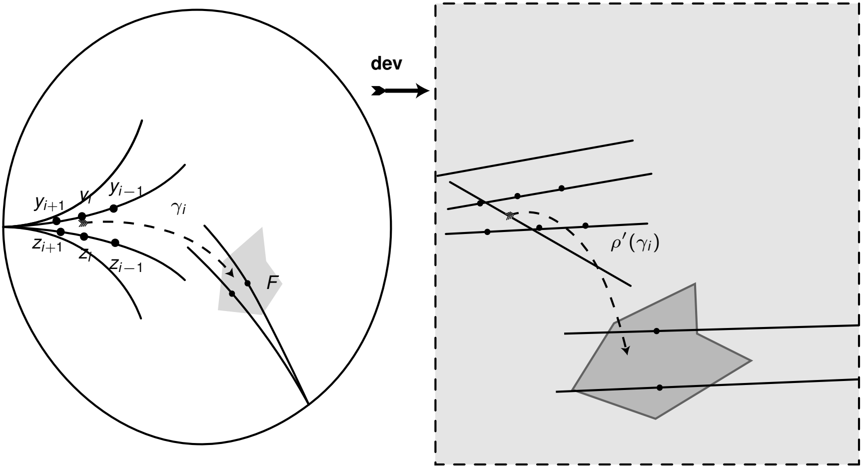

Figure: The proof of Theorem 1. Here γ_i is multiplied by an element to make the figure look better.

- We can choose two leaves l_y and l_z in $\mathcal{R}_p$ $y, z \in \mathbf{U}\tilde{M}$, so that $\tilde{\mathcal{I}_\infty}(l_y)$ and $\tilde{\mathcal{I}_\infty}(l_z)$ are in distinct subspaces $A^{0-}_{l_y}$ and $A^{0-}_{l_z}$ by Proposition 3.
- The following contradiction proves Theorem 1.

Proposition 4

There are no two leaves l_y and l_z in $\mathcal{R}_p$ for $y, z \in \mathbf{U}\tilde{M}$ so that so that $\tilde{\mathcal{I}_\infty}(l_y)$ and $\tilde{\mathcal{I}_\infty}(l_z)$ are in distinct subspaces $A^{0-}_{l_y}$ and $A^{0-}_{l_z}$

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Proof begins

Suppose not. Also, under $\tilde{\pi}_M$, l_y and l_z respectively go to geodesics ending at p. We assume that an infinite order deck transformation γ acts on the axis containing l_y and fixes p.
$A^{0-}_{\phi_t(y)}$ is a fixed affine subspace independent of t, and $\rho'(\gamma)$ acts on $A^{0-}_{\phi_t(y)}$.

Pulling-back argument

- $A^0_{\phi_t(z)}$ contains I_z and $\mathbb{V}^-(\phi_t(z))$ is independent of t since they are parallel under the flat connection.

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- $A^0_{\phi_t(z)}$ contains l_z and $\mathbb{V}^-(\phi_t(z))$ is independent of t since they are parallel under the flat connection.
- Choose $y_i \in l_y$ so that $y_i = \phi_{t_i}(y)$, and $z_i \in l_z$ so that $z_i = \phi_{t_i}(z)$ where $t_i \to \infty$ as $i \to \infty$. Denote by

$$y_i' := \tilde{\mathcal{I}_\infty}(y_i) \text{ and } z_i' := \tilde{\mathcal{I}_\infty}(z_i) \text{ in } \mathbb{A}^n.$$

- Since $\langle\gamma\rangle$ acts on the axis containing l_y, $\gamma_i(y_i)$ is in a compact subset F of $\mathbf{U}\tilde{M}$ for a sequence $\gamma_i = \gamma^{-j_i}$ with j_i going to infinity. $\rho'(\gamma_i)(y_i')$ is in a compact subset of $\mathbb{A}^n$ for $y_i' = \tilde{\Pi}_M(y_i)$.

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- Choose a subsequence so that

$$\rho'(\gamma_i)(y_i') \to y_\infty' \text{ for a point } y_\infty' \in \mathbb{A}^n. \tag{8}$$

- Since s_∞ is continuous by Proposition 2, we obtain

$$d_{\mathbb{A}^n/\Gamma}(\tilde{\mathcal{I}_\infty}(y_i), \tilde{\mathcal{I}_\infty}(z_i)) \to 0. \tag{9}$$

- Since γ_i is an isometry of $d_{\mathbb{A}^n}$,

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Lemma 5

$A^{0-}_{l_z}$ is affinely parallel to $A^{0-}_{l_y}$.

Proof.

Otherwise, we can show $\rho(\gamma_i)(A^{0-}_{l_z}) = A^{0-}_{\gamma_i(z_i)}$ does not converge to $A^{0}_{l_y}$. But $d_M(\gamma_i(z_i), \gamma_i(y_i)) \to 0$. □

- Also the sequence of the Hausdorff distance between

$$A^{0-}_{\gamma_i(z_i)} = \rho'(\gamma_i)(A^{0-}_{l_z}) \text{ and } A^{0-}_{\gamma_i(y_i)} = \rho'(\gamma_i)(A^{0-}_{l_y})$$

  is going to 0.

- Let $\vec{v}$ denote the vector in the direction of $\mathbb{V}_+(y_i)$ going from y_i to $A^{0-}_{l_z}$, independent of y_i. Then for the linear part A_{γ_i} of the affine transformation γ_i,

$$\|v_i' := A_{\gamma_i}(\vec{v})\|^E_n \to \infty.$$

- Hence affine subspaces

$$A^{0-}_{\gamma_i(z_i)} = \rho'(\gamma_i)(A^{0-}_{l_z}) \text{ and } A^{0-}_{\gamma_i(y_i)} = \rho'(\gamma_i)(A^{0-}_{l_y})$$

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- See following diagram as a proof.

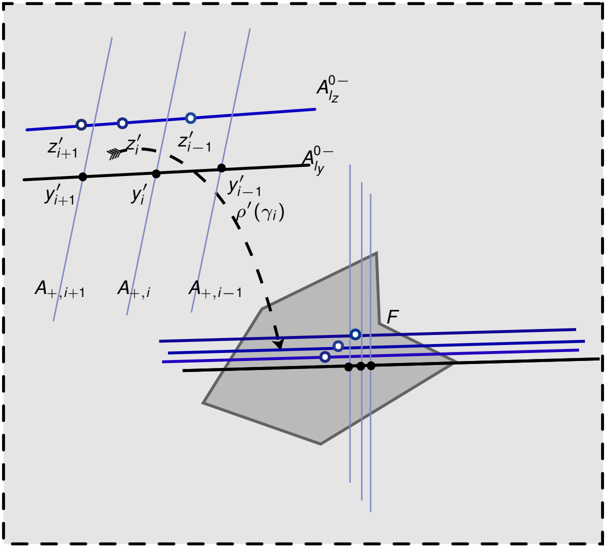

Figure: The proof of Theorem 1

III: Generalization without negative curvature conditions

- Assume that $\tilde{M}$ is Gromov hyperbolic.
- A *complete isometric geodesic* in $\tilde{M}$ is a geodesic that is an isometry of $\mathbb{R}$ into $\tilde{M}$ equipped with a Riemannian metric. A *complete isometric geodesic* in M is a geodesic that lifts to a complete isometric geodesic in $\tilde{M}$.

III: Generalization without negative curvature conditions

- Assume that $\tilde{M}$ is Gromov hyperbolic.
- A *complete isometric geodesic* in $\tilde{M}$ is a geodesic that is an isometry of $\mathbb{R}$ into $\tilde{M}$ equipped with a Riemannian metric. A *complete isometric geodesic* in M is a geodesic that lifts to a complete isometric geodesic in $\tilde{M}$.
- We consider the subset of $\mathbf{U}M$ where complete isometric geodesics pass. We denote this set by $\mathbf{UC}M$, and call it the *complete-isometric-geodesic unit-tangent bundle*.
- The inverse image in $\mathbf{U}\tilde{M}$ is denoted by $\mathbf{UC}\tilde{M}$. Clearly, $\mathbf{UC}M$ is compact and $\mathbf{UC}\tilde{M}$ is locally compact. However, $\tilde{\pi}_M(\mathbf{UC}\tilde{M})$ may be a proper subset of $\tilde{M}$.
- Now we define partial hyperbolicity over $\mathbf{UC}M$ only.

Generalization of Theorem 1

Theorem 6

Let M be a closed complete special affine n-manifold. Then the linear part of a holonomy homomorphism ρ is not a partially hyperbolic representation in a bundle sense.

Generalization of Theorem 1

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Let M be a closed complete special affine n-manifold. Then the linear part of a holonomy homomorphism ρ is not a partially hyperbolic representation in a bundle sense.

- Partial hyperbolicity $\longrightarrow$ P-Anosov for $k = \dim \mathbb{V}_+$.
- Now, by Kapovich-Leeb-Porti, $\pi_1(M)$ is hyperbolic.
- Hence, $\tilde{M}$ is Gromov hyperbolic by Svarc-Milnor.

Let p be a point of the Gromv boundary $\partial_\infty \tilde{M}$. Let $\mathcal{R}_p$ denote the union of complete isometric geodesics in $\mathbf{UC}\tilde{M}$ mapping to complete isometric geodesics in $\tilde{M}$ ending at p.

Let p be a point of the Gromv boundary $\partial_\infty \tilde{M}$. Let $\mathcal{R}_p$ denote the union of complete isometric geodesics in $\mathbf{UC}\tilde{M}$ mapping to complete isometric geodesics in $\tilde{M}$ ending at p.

Proposition 7

Let M be a closed manifold with a Riemannian metric. Suppose that $\pi_1(M)$ is hyperbolic. Let $p \in \partial_\infty \tilde{M}$. Then $\pi_{\tilde{M}}(\mathcal{R}_p)$ is C-dense in $\tilde{M}$.

Proposition 8 (Modification)

There is a section $s_\infty : \mathbf{UC}M \to \mathbb{A}^n_{\rho'}$*, homotopic to the developing section* $s|\mathbf{UC}M$ *in the* C^0*-topology with the following conditions*:

- $\nabla_\phi s_\infty$ *is in* $\mathbb{V}_0(x)$ *for each* $x \in \mathbf{UC}M$.
- $d_{\mathbb{A}^n_{\rho'}}(s(x), s_\infty(x))$ *is uniformly bounded for every* $x \in \mathbf{UC}M$.
- $d_{\mathbb{A}^n}(\tilde{\mathcal{I}}(x), \tilde{\mathcal{I}_\infty}(x))$ *is uniformly bounded for* $x \in \mathbf{UC}\tilde{M}$.
- $\tilde{\mathcal{I}_\infty} : \mathbf{UC}\tilde{M} \to \mathbb{A}^n$ *is properly homotopic to* $\tilde{\mathcal{I}}$ *and is coarsely Lipschitz.*

Now, the proof of Theorem 6 proceeds similar to that of Theorem 1. However, we need some rough geometry ideas.

Theorem 9 (Choi-Kapovich)

Suppose that M is a closed complete affine manifold covered by an affine space $\tilde{M} = \mathbb{A}^n$ with the Riemannain metric d_M induced from that of M. Let L be an affine subspace of lower-dimension of $\tilde{M}$. Then $\tilde{M}$ is not a C-neighborhood $N_C(L)$ of L.

Proof.

Follows from the following two theorems. □

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Proof.

Follows from the following two theorems. □

Proposition 10 (Choi-Kapovich)

Let M and L be as above. Then L with induced path-metric d_L is uniformly properly embedded in $\tilde{M} = \mathbb{A}^n$.

Proof.

Just need to show if two points are of bounded distance under d_M, the path-distance in L cannot go to infinity. Here, we may assume that one point is in a fundamental domain using deck transformations. □

Theorem 11

Let M and L be as above. Then L is uniformly contractible with respect to the path metric on L induced from d_M.

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Proof.

Any sphere map $f : S^i \to L$ with a d_M-diameter C may be moved by a deck transformation γ to a one passing a fundamental domain F of $\mathbb{A}^n$. Hence, a Euclidean ball B_R of some radius contains the image of $\gamma \circ f$. Here R depends only on C. Now, $B_R \subset B^M_{R'}$ for a d_M-ball $B^M_{R'}$ for a radius R' depending only on R. Hence, f is homotopic to a point inside $\gamma^{-1}(B^M_{R'})$ for R' depending only on C. □

Recall $H^n_C(X) := \varinjlim H^n(X, X - K)$ for K a compact subset of X. For $X = R^n$, $H^n_C(X) = \mathbb{Z}$.

Theorem 12 (Kapovich)

Let X be an open n-manifold that is a contractible δ-hyperbolic complete Riemannian metric space with the path metric d_X. Let U be a uniformly properly embedded open cell with the induced path-metric so that U is uniformly contractible and coarsely equivalent to X. Then U must have the topological dimension n.

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Proof.

There is an inclusion map $f : U \to X$ and its rough inverse map $g : X \to U$. We may assume that both are continuous. Then $f \circ g$ is homotopic to identity by a bounded continuous homotopy. Then $g_* \circ f_* : H^n_C(X) \to H^n_C(X)$ is an isomorphism. Since $H^n_C(U)$ has to be nonzero, $\dim U = \dim X$. □

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