Lens-shaped totally geodesic ends of convex real projective manifolds

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Abstract

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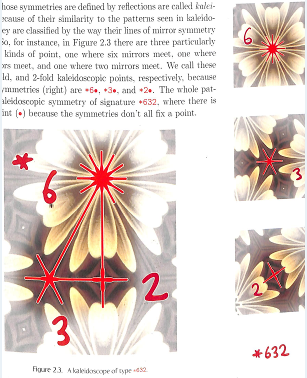
- We also discuss the relationship to the globally hyperbolic space-times in flat Lorentz geometry.
- The lens-shaped radial ends which are dual to lens-shaped totally geodesic ends.

Outline

- Introduction: orbifolds, geometric structures, projective, affine, and hyperbolic geometry, real projective structures
- Main result:
  - Totally geodesic ends
  - Asymptotically nice action
  - Outline of proof
  - Globally hyperbolic spacetime
- Duality and R-ends

Orbifolds

- By an *n*-dimensional orbifold is a space modelled on finite quotients of open sets (with some compatibility conditions.)
- Let P be a convex polyhedron and we silver each side where the angles are of form π/n: *Coxeter orbifolds*.



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- Examples: a square with silvered edges, a triangular orbifold (Conway's picture)
- A good orbifold: M/Γ where Γ is a discrete group with a properly discontinuous action.

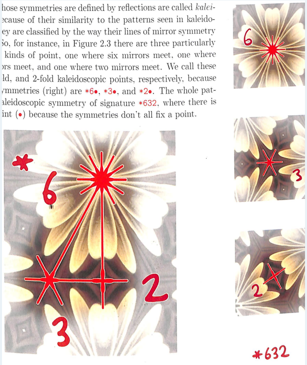

Figure 2.3. A kaleidoscope of type *632.

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(G, X)-structure on orbifolds

Given a manifold or orbifold M, we locally model M by open subsets of X or their finite quotients pasted by elements of G. The compatibility class of the atlas of charts is a (G, X)-*structure* on M.

Projective, affine geometry

- $\mathbb{R}P^n = P(\mathbb{R}^{n+1}) = (\mathbb{R}^{n+1} - \{O\})/\sim$ where $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{O\}$.
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- $\mathbb{R}P^n - \mathbb{R}P^{n-1}_\infty$ is an affine space $\mathbb{A}^n$ where the group of projective automorphisms of $\mathbb{A}^n$ is exactly $Aff(\mathbb{A}^n)$.

$$\mathbb{A}^n \hookrightarrow \mathbb{R}\mathbb{P}^n, Aff(\mathbb{A}^n) \hookrightarrow \mathrm{PGL}(n+1, \mathbb{R}).$$

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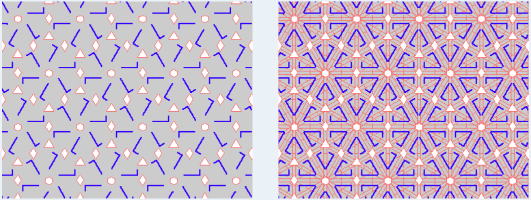

Figure: Wall-paper groups 16 and 17.

Hyperbolic geometry

- $\mathbb{R}^{1,n}$ with Lorentzian metric $q(\vec{v}) := -x_0^2 + x_1^2 + \cdots + x_n^2$.
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- The cone $q < 0$ corresponds to the convex open n-ball in $B^n \hookrightarrow \mathbb{A}^n \subset \mathbb{R}P^n$ correspond to $\mathbb{H}^n$ in a one-to-one manner.
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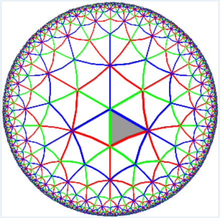

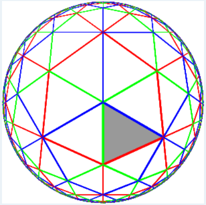

Figure: The triangle group $D^2(3,3,4)$ in the Poincare and Klein models by Bill Casselman.

Real projective structures on orbifolds

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- The quotient D/Γ for a properly acting discrete group $\Gamma \subset \mathbf{Aut}(D)$ is called a convex real projective orbifold.
- If D is properly convex, then D/Γ is called a *properly convex real projective orbifold*.

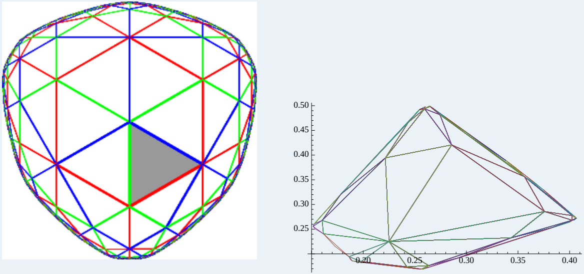

Figure: The developing images of convex $\mathbb{R}P^n$-structures on 2-orbifolds deformed from hyperbolic ones: $D(3,3,4)$, $S^2(3,3,5)$

Oriented real projective space

- We can double cover to obtain $\mathbb{S}^n \to \mathbb{RP}^n$. The group of projective automorphisms is $\mathrm{SL}_\pm(n+1,\mathbb{R})$.
- Here, $\mathbb{A}^n$ corresponds to an open hemisphere with ideal boundary $\mathbb{S}^{n-1}_\infty$.
- Every convex real projective structure on a closed orbifold can be thought of as being of form Ω/Γ where Ω is a convex domain in an open hemisphere. Hence, the holonomy of any closed curve can be uniquely chosen as an element of $\mathrm{SL}_\pm(n+1,\mathbb{R})$.
- The space of oriented hyperplanes in $\mathbb{S}^n$ forms the dual space $\mathbb{S}^{n*}$.

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A *lens neighborhood* of Σ_E is a one-sided compact neighborhood of totally geodesic ideal boundary component in an ambient real projective orbifold covered by a lens.

End fundamental group and pseudo-ends

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- In general these are somewhat related to the recent work of Cooper, Long, Tillman, Ballas, and Leitner.

Properly convex affine action

- An *affine deformation* of linear $\hat{h} : \Gamma \to \mathrm{GL}_{\pm}(n, \mathbb{R})$ is given by a cocycle

$$\vec{b} : g \to \vec{b}_g \in \mathbb{R}^n, g \in \Gamma$$

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- When Γ fixes a point or equivalently $\vec{b} \sim 0$, U can be a cone. U can also be a properly convex domain with smooth boundary.

Asymptotically nice action

Definition 3.1 (AS-hyperspace).

A sharply supporting hyperspace P at $x \in \mathrm{bd}\Omega$ is *asymptotic* to U if there are no other sharply supporting hyperplane P' at x so that $P' \cap \mathbb{A}^n$ separates U and $P \cap \mathbb{A}^n$. P is *asymptotic* to U.

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A properly convex affine action of Γ is said to be *asymptotically nice* if

- Γ acts on U in $\mathbb{A}^n$ with boundary in $\mathrm{Cl}(\Omega) \subset \mathbb{S}^{n-1}_\infty$,
- Γ acts on a compact subset $J := \{H | H \text{ is an AS- hyperspace in } \mathbb{S}^n \text{ at } x \in \mathrm{bd}\Omega, H \not\subset \mathbb{S}^{n-1}_\infty\}$, requiring that every sharply supporting $(n-2)$-dimensional space of Ω in $\mathbb{S}^{n-1}_\infty$ is contained in an element of J.

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As a consequence, for any sharply supporting $(n-2)$-dimensional space Q of Ω, the set

$$H_Q := \{H \in J | H \supset Q\}$$

is compact and bounded away from $\mathrm{bd}\mathbb{A}^n$ in the Hausdorff metric $\mathbf{d}_H$.

Affine action

- For each element of $g \in \Gamma_{\tilde{E}}$, represented as a $\det = \pm 1$ matrix,

$$h(g) = \begin{pmatrix} \frac{1}{\lambda_{\tilde{E}}(g)^{1/n}} \hat{h}(g) & \vec{b}_g \\ \vec{0} & \lambda_{\tilde{E}}(g) \end{pmatrix} \tag{1}$$

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- In the affine coordinates, it is of the form

$$x \mapsto \frac{1}{\lambda_{\tilde{E}}(g)^{1+\frac{1}{n}}} \hat{h}(g) x + \frac{1}{\lambda_{\tilde{E}}(g)} \vec{b}_g. \tag{2}$$

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where $\vec{b}_g$ is an $n \times 1$-vector and $\hat{h}(g) \in \mathrm{SL}_\pm(n, \mathbb{R})$ and $\lambda_{\tilde{E}}(g) > 0$.

- For $\lambda_1(g)$ the max-norm of the eigenvalues of $g \in \Gamma_E$, if there exists a uniform constant $C > 1$ so that

$$C^{-1}\mathrm{length}_\Omega(g) \leq \log \frac{\lambda_1(g)}{\lambda_{\tilde{E}}(g)} \leq C\mathrm{length}_\Omega(g), \quad g \in \Gamma_{\tilde{E}} - \{\mathrm{I}\}, \quad (3)$$

then Γ satisfies the *uniform middle eigenvalue condition* with respect to the boundary hyperspace.

Asymptotic niceness

Theorem 1.

We assume that Γ is a discrete affine group dividing an open properly convex domain Ω in $\mathrm{bd}\mathbb{A}^n$. Suppose that Γ satisfies the uniform middle-eigenvalue condition with respect to $\mathrm{bd}\mathbb{A}^n$. Then

- *Γ is asymptotically nice with respect to a properly convex open domain U, and*
- *any open set U' satisfying the properties of U has the AS-hyperspace at each point of $\mathrm{bd}\Omega$ is the same as that of U.*

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Remark

Below, I give a proof for hyperbolic Γ or strictly convex Ω. Here, $\hat{\mathbb{U}}\Omega = \mathbb{U}\Omega$.

Hyperbolic and nonhyperbolic Γ: dichotomy by Koszul, Vey, Benoist

Assume Ω/Γ is a closed orbifold.

- - Γ is hyperbolic $\leftrightarrow$
  - Ω is strictly convex and have $C^{1,\varepsilon}$-boundary $\rightarrow$
  - each infinite order elements are positive proximal. (Benoist)

  (Here unique supporting hyperplane for each $x \in \mathrm{bd}\Omega$.)

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Assume Ω/Γ is a closed orbifold.

- Otherwise, $\mathrm{bd}\Omega$ may have flats and infinite order elements are only positive semiproximal.
  - decomposable: Here, we may have decomposition $\mathrm{Cl}(\Omega) = K_1 * \cdots * K_m$ and Γ is virtually a subgroup of $\mathbb{Z}^{m-1} \times \Gamma_1 \times \cdots \times \Gamma_m$ with $\mathbb{Z}^{m-1}$ in the center and positive diagonalizable.
  - nondecomposible: No virtual infinite center case: fairly nice properties.

(here, many supporting hyperplanes for each $x \in \mathrm{bd}\Omega$.)

Generalization of unit tangent bundles

We generalize $\mathbb{U}\Omega$ to the *augmented unit tangent bundle*

$$\begin{aligned}\hat{\mathbb{U}}\Omega := \{(\vec{x}, H_a, H_r) | \ \vec{x} \in \mathbb{U}\Omega & \text{ is a direction vector at a point} \\ & \text{of a maximal oriented geodesic } l_{\vec{x}} \text{ in } \Omega, \\ H_a & \text{ is a sharply supporting hyperspace at the starting point of } l_{\vec{x}}, \\ H_r & \text{ is a sharply supporting hyperspace at the ending point of } l_{\vec{x}}.\} \end{aligned} \quad (4)$$

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- $\tilde{\mathbf{A}} := \mathbb{U}\Omega \times \mathbb{A}^n$ with action

$$g(x, \vec{u}) = (g(x), h(g)\vec{u}) \text{ for } g \in \Gamma, x \in \mathbb{U}\Omega, \vec{u} \in \mathbb{A}^n.$$

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  The quotient space **A** fibers over $\mathbb{U}\Omega/\Gamma$ with fiber $\cong \mathbb{A}^n$.
- $\tilde{\mathbf{V}} := \mathbb{U}\Omega \times \mathbb{R}^n$ and take the quotient under the diagonal action:

$$g(x,\vec{u}) = (g(x), \mathscr{L}(h(g))\vec{u}) \text{ for } g \in \Gamma, x \in \mathbb{U}\Omega, \vec{u} \in \mathbb{R}^n.$$

  The quotient space **V** fibers over $\mathbb{U}\Omega/\Gamma$ with fiber $\mathbb{R}^n$. (A fiberwise tangent bundle of **A**)

Proximal flows

- the unit tangent space $\mathbb{U}\Omega := T\Omega - O/\sim$, affine space $\mathbb{A}^n$ and associated vector space version $\mathbb{R}^n$.
- $\tilde{\mathbf{A}} := \mathbb{U}\Omega \times \mathbb{A}^n$ with action

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  The quotient space **V** fibers over $\mathbb{U}\Omega/\Gamma$ with fiber $\mathbb{R}^n$. (A fiberwise tangent bundle of **A**)
- There is a flat connection $\nabla^{\mathbf{V}}$ on **V**.

- There are fiberwise metrics denoted by $\|\cdot\|_{\mathrm{fiber}}$ on **A** and **V** obtainable by partition of unity.
- There is a flow $\Phi_t : \mathbf{A} \to \mathbf{A}$, $t \in \mathbb{R}$ acting parallel way on the fibers and acting as a flow on $\mathbb{U}\Sigma_E$.
- Also a flow $\mathscr{L}(\Phi_t) : \mathbf{V} \to \mathbf{V}$.

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- Also a flow $\mathscr{L}(\Phi_t) : \mathbf{V} \to \mathbf{V}$.
- The aim is to find a section $s : \mathbb{U}\Sigma_E \to \mathbb{A}^n$ "neutral" under flow Φ. That is the geodesics goes to an arc varying in only neutral directions.

Decomposition of **V**.

- For each vector $\vec{u} \in \mathbb{U}\Omega$, the oriented geodesic l ending at $\partial_+ l, \partial_- l \in \mathrm{bd}\Omega$. correspond to the 1-dim vector subspaces $\mathbf{V}_+(\vec{u})$ and $\mathbf{V}_-(\vec{u})$ in $\mathbb{R}^n$.

Decomposition of $\mathbf{V}$.

- There exists a unique pair of sharply supporting hyperspaces H_+ and H_- in $\mathrm{bd}\mathbb{A}^n$ at $\partial_+ l$ and $\partial_- l$. We denote by $H_0 = H_+ \cap H_-$, a codimension 2 great sphere in $\mathrm{bd}\mathbb{A}^n$ and corresponds to a subspace $\mathbf{V}_0(\vec{u})$ of codimension-two in $\mathbf{V}$.

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- For each vector $\vec{u}$, we obtain the C^0-decomposition of V as $\mathbf{V}_+(\vec{u}) \oplus \mathbf{V}_0(\vec{u}) \oplus \mathbf{V}_-(\vec{u})$ and form the subbundles $\tilde{\mathbf{V}}_+, \tilde{\mathbf{V}}_0, \tilde{\mathbf{V}}_-$ over $\mathbb{U}\Omega$ where $\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_+ \oplus \tilde{\mathbf{V}}_0 \oplus \tilde{\mathbf{V}}_-$.

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Since the Γ-action and Φ preserves the decomposition of $\tilde{\mathbf{V}}$, we obtain the bundle decomposition

$$\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_0 \oplus \mathbf{V}_- \tag{5}$$

invariant under the flow $\mathscr{L}(\Phi)_t$.

Flow properties of $\mathscr{L}(\Phi_t)$

By the uniform middle-eigenvalue condition, **V** satisfies the following properties:

- the flat linear connection $\nabla^{\mathbf{V}}$ on **V** is bounded with respect to $\|\cdot\|_{\text{fiber}}$.

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$$\left\|\mathscr{L}(\Phi_t)(\vec{v})\right\|_{\mathrm{fiber}} \geq \frac{1}{C}\exp(kt)\left\|\vec{v}\right\|_{\mathrm{fiber}} \text{ as } t \to \infty \tag{6}$$

  for $\vec{v} \in \mathbf{V}_+$ and

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for $v \in \mathbf{V}_-$.
- Intuitively, these are just from the fact that as one goes along the flow the forward vector must become larger and larger in order for it to be in the same size as in $\|\cdot\|_{\text{fiber}}$.

Proof of the flow properties: $\mathbf{V}_-$-part only

- Let F be a compact fundamental domain of Ω. We let $y_i = \Phi_{t_i}(x_i)$, $t_i \to \infty$, for x_i is some compact set K in $\mathbb{U}\Omega$. Let l_i denote the complete geodesic containing x_i, y_i.

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- Find a deck transformation g_i so that $g_i(y_i) \in F$ and g_i acts on the line bundle $\tilde{\mathbf{V}}_-$ by the linearization of the matrix of form of (1):

$$\mathscr{L}(g_i) : \mathbf{V}_- \to \mathbf{V}_- \text{ given by}$$
$$(y_i, \vec{v}) \to (g_i(y_i), \mathscr{L}(g_i)(\vec{v})) \text{ where}$$
$$\mathscr{L}(g_i) = \frac{1}{\lambda_{\tilde{E}}(g_i)^{1+\frac{1}{n}}} \hat{h}(g_i) : \mathbf{V}_-(y_i) = \mathbf{V}_-(x_i) \to \mathbf{V}_-(g_i(y_i)). \tag{8}$$

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(Goal) We will show $\{(g_i(y_i), \mathscr{L}(g_i)(\vec{v}_{-,i}))\} \to 0$ under $\|\cdot\|_{\text{fiber}}$. This will complete the proof since g_i acts as isometries on $\mathbf{V}_-$ with $\|\cdot\|_{\text{fiber}}$.

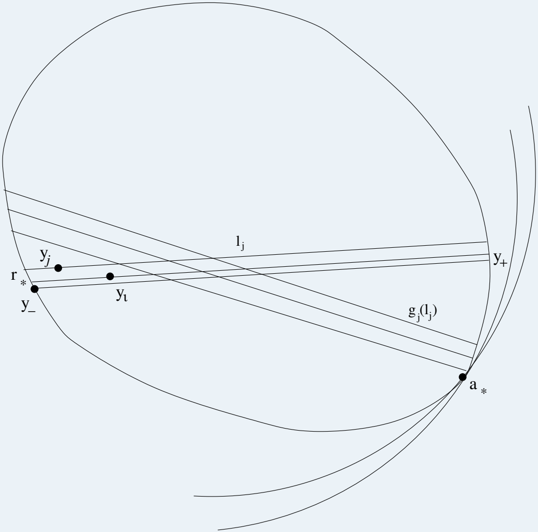

Figure: Pulling back arguments.

Proof continues

Let $\|\cdot\|_E$ denote the standard Euclidean metric of $\mathbb{R}^n$. Let a_i, r_i denote the attracting and repelling fixed poins of g_i.

- Since $\Pi_\Omega(y_i) \to y_-$, $\Pi_\Omega(y_i)$ is also uniformly bounded away from a_i and the tangent sphere $\mathbb{S}_i^{n-1}$ at a_i.

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- Since $((\vec{v}_{-,i})) \to y_-$, the vector $\vec{v}_{-,i}$ has the component $\vec{v}_i^p$ parallel to r_i and the component $\vec{v}_i^S$ in the direction of $\mathbb{S}_i^{n-2}$ where $\vec{v}_{-,i} = \vec{v}_i^p + \vec{v}_i^S$.

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- Since $r_i \to r_* = y_-$ and $((\vec{v}_{-,i})) \to y_-$, we obtain $\|\vec{v}_i^S\|_E \to 0$ and that
$$\frac{1}{C} < \|\vec{v}_i^p\|_E < C$$
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  for some constant $C > 1$.
- g_i acts by preserving the directions of $\mathbb{S}_i^{n-2}$ and r_i.

Since $\{g_i(((\vec{v}_{-,i})))\}$ converging to y', $y' \in \mathrm{bd}\Omega$, is bounded away from $\mathbb{S}_i^{n-2}$ uniformly, we obtain that

- considering the homogeneous coordinates

$$\left(\left(\mathscr{L}(g_i)(\vec{v}_i^S) : \mathscr{L}(g_i)(\vec{v}_i^p)\right)\right)$$

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Since r_i is a repelling fixed point of g_i and $\left\|\vec{v}_i^p\right\|_E$ is uniformly bounded above, $\{\mathscr{L}(g_i)(\vec{v}_i^p)\} \to 0$ by (3) and $\mathrm{length}_\Omega(g_i) \to \infty$. $\{\mathscr{L}(g_i)(\vec{v}_i^p)\} \to 0$ implies $\{\mathscr{L}(g_i)(\vec{v}_i^S)\} \to 0$ for $\|\cdot\|_E$. Hence, we obtain $\left\|\mathscr{L}(g_i)(\vec{v}_{-,i}))\right\|_E \to 0$.

Neutralized sections

$\mathbf{A} := \mathbb{U}\Omega \times \mathbb{A}^n/\Gamma$. A section $s : \mathbb{U}\Omega/\Gamma \to \mathbf{A}$ is *neutralized* if

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Lemma 2.

A neutralized section $s_0 : \mathbb{U}\Omega/\Gamma \to \mathbf{A}$ exists. This lifts to a map $\tilde{s}_0 : \mathbb{U}\Omega \to \tilde{\mathbf{A}}$ so that $\tilde{s}_0 \circ \gamma = \gamma \circ \tilde{s}_0$ for each γ in Γ acting on $\tilde{\mathbf{A}} = \mathbb{U}\Omega \times \mathbb{A}^n$.

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Proof.

We decompose

$$\nabla_\phi^{\mathbf{V}}(s) = \nabla_\phi^{\mathbf{V}_+}(s) + \nabla_\phi^{\mathbf{V}_0}(s) + \nabla_\phi^{\mathbf{V}_-}(s) \in \mathbf{V}$$

so that $\nabla_\phi^{\mathbf{V}_\pm}(s) \in \mathbf{V}_\pm$ and $\nabla_\phi^{\mathbf{V}_0}(s) \in \mathbf{V}_0$ hold. The integrals converge to smooth functions over $\mathbb{U}\Omega/\Gamma$: $s_0 = s + \int_0^\infty (D\Phi_t)_*(\nabla_\phi^{\mathbf{V}_-}(s))dt - \int_0^\infty (D\Phi_{-t})_*(\nabla_\phi^{\mathbf{V}_+}(s))dt$ is a continuous section and $\nabla_\phi^{\mathbf{V}}(s_0) = \nabla_\phi^{\mathbf{V}_0}(s_0) \in \mathbf{V}_0$ as shown by Goldman-Labourie-Margulis [2]. □

Finding supporting hyperspaces

- Let $N_2(\mathbb{A}^n)$ denote the space of codimension-two affine subspaces of $\mathbb{A}^n$.
- Each geodesic goes into a neutral affine subspace of codimension two in $\mathbb{A}^n$. There exists a continuous function $\hat{s} : \mathbb{U}\Omega \to N_2(\mathbb{A}^n)$ equivariant with respect to Γ-actions.

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- $$\bar{s} : \Lambda^* = \mathrm{bd}\Omega \times \mathrm{bd}\Omega - \Delta \to N_2(\mathbb{A}^n)$$

  is continuous and equivariant wrt the Γ-actions.

Theorem 3 (Existence of U).

- *Let Γ have an affine action on the affine subspace $\mathbb{A}^n$, $\mathbb{A}^n \subset \mathbb{S}^n$, acting properly and cocompactly on a properly convex domain Ω in $\mathrm{bd}\mathbb{A}^n$.*
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Proof.

- We form an affine subspace of codim 1 for each oriented (augmented) geodesic l by taking an affine span of the backward (augmented) endpoint of l and the affine subspace $\hat{s}(\vec{l}_g)$ in $N_2(\mathbb{A}^n)$.
- This affine subspace depends only on the backward endpoint of l.
- Then we form an affine half-space by taking the component containing Ω in the boundary.

□

Theorem 4 (Uniqueness of the set of AS-hyperplanes).

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- *Suppose that* Ω/Γ *is a closed* $(n-1)$*-dimensional orbifold.*
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The action is expansive on the space of parallel AS-hyperplanes. □

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Proof of Theorem 1.

Theorems 3 and 4 prove it. □

Globally hyperbolic spacetime

- When the linear holonomy is convex cocompact in $SO(n-1,1)$, Ω is a standard ball in $\mathbb{S}^{n-1}_\infty$ and Thierry Barbot showed that there exists Γ acting on properly convex domain U in $\mathbb{A}^n$ with $\mathrm{Cl}(U) \cap \mathbb{S}^{n-1}_\infty = \mathrm{Cl}(\Omega)$.

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- We can form the maximal globally hyperbolic space-time in $\mathbb{R}^{n-1,1}$. One can find a foliation by Cauchy hypersurfaces. (For compact Ω/Γ, these were done by Geroch in 1970s, and the convex domain U can be chosen to be bounded by an affine sphere by Loftin and Labourie as shown in late 90s but with no AS niceness.)

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- We can form the maximal globally hyperbolic space-time in $\mathbb{R}^{n-1,1}$. One can find a foliation by Cauchy hypersurfaces. (For compact Ω/Γ, these were done by Geroch in 1970s, and the convex domain U can be chosen to be bounded by an affine sphere by Loftin and Labourie as shown in late 90s but with no AS niceness.)
- Barbot's technique is that of Mane-Sullivan. One extends the action to parallel null-planes and use contraction properties to obtain the Sullivan stability.

Duality and R-ends

- Given a properly convex affine action of Γ acting on

$$U \subset \mathbb{A}^n \subset \mathbb{S}^n \text{ and } \Omega \subset \mathbb{S}^{n-1}_\infty,$$

the dual group Γ^* will act on $\mathbb{S}^{n*}$ fixing a pair of points p and p_- in $\mathbb{S}^{n*}$ dual to $\mathbb{S}^{n-1}_\infty$ with two orientations.

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- Γ^* acts on a union of segments $\mathscr{T}_p$ from p to p_- and the spaces of whose directions form a properly convex domain projectively diffeomorphic to Ω^* in $\mathbb{S}^{n-1}_p$.
- There is a fibration $\partial T_p - \{p, p_-\} \to \partial\Omega^*$ with fiber a line from p to p_-.
- The uniform middle eigenvalue condition in this case implies the existence of the continuous Γ^*-invariant section $\partial\Omega \to \partial\mathscr{T}_p - \{p, p_-\}$.

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  the dual group Γ^* will act on $\mathbb{S}^{n*}$ fixing a pair of points p and p_- in $\mathbb{S}^{n*}$ dual to $\mathbb{S}^{n-1}_\infty$ with two orientations.
- Oriented hyperspaces meeting $\mathbb{S}^{n-1}_\infty$ in a fixed codimension 2 subspace disjoint from Ω form a segment from p to p_- in $\mathbb{S}^{n*}$.
- Γ^* acts on a union of segments $\mathscr{T}_p$ from p to p_- and the spaces of whose directions form a properly convex domain projectively diffeomorphic to Ω^* in $\mathbb{S}^{n-1}_p$.
- There is a fibration $\partial T_p - \{p, p_-\} \to \partial\Omega^*$ with fiber a line from p to p_-.
- The uniform middle eigenvalue condition in this case implies the existence of the continuous Γ^*-invariant section $\partial\Omega \to \partial\mathscr{T}_p - \{p, p_-\}$.
- This gives a lens-domain in $\mathscr{T}_p$ disjoint from $\{p, p_-\}$.

Lens-shaped R-ends

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- Such a p-end neighborhood has a foliation by radial segments, and the corresponding end is called a *lens-shaped radial end*.
- Again, the uniform middle eigenvalue condition for the end holonomy group Γ is equivalent to the existence of generalized lens-shaped radial end-neighborhood.

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