Algebraic Topology I: Midterm Exam

Justify your answers fully.

1. Sow that there are no retractions $r : X \to A$ in the following cases:
   (a) (10 pts.) $X = R^3$ with A any subspace homeomorphic to S^1.
   (b) (10 pts.) $X = D^2 \vee D^2$ with A equal to its boundary $S^1 \vee S^1$.
   (c) (10 pts.) X a Mobius band and A its boundary circle.

2. Let $f : X \to Y$ be a homotopy equivalence.
   (a) (10 pts.) Show that f induces a bijection between the set of components of X to the set of components of Y.
   (b) (10 pts.) Show that f induces a bijection between the set of path-components of X to the set of path-components of Y.
   (c) (10 pts.) Show that f restricts to a homotopy equivalence between a path-component of X to a corresponding path-component of Y.

3.
   (a) (10 pts.) Find a nonregular 4-fold cover of a bouquet of two circles using the fundamental groups. Find also a regular one.
   (b) (10 pts.) Show that a punctured torus is homotopy equivalent to the bouquet of two circles.
   (c) (10 pts.) Find a nonregular 4-fold cover of a punctured torus using the fundamental groups. Also, find a regular one.

4. (30 pts.) Let X be the space obtained from the two-sphere S^2 and an interval by identifying the endpoints with a single point of S^2. Compute the fundamental group of X and classify the covering spaces of X up to covering equivalences.

5. Let $p : \tilde{X} \to X$ be a universal covering map of X.
   (a) (10 pts.) Show that X is homeomorphic to $\tilde{X}/G$ for the deck transformation group G.
   (a) (10 pts.) Show that for any subgroup H of G, the induced map $\tilde{X}/H \to X$ is a covering map.
   (b) (10 pts.) Show that a covering map $p_Y : Y \to X$ of X is isomorphic to $\tilde{X}/H \to X$ for some group H, and determine H from knowing p_Y.