Algebraic Topology I: Midterm Exam: Brief Answers

Justify your answers fully.

1. Sow that there are no retractions $r : X \to A$ in the following cases:
   (a) (10 pts.) $X = R^3$ with A any subspace homeomorphic to S^1.
   Given a retraction $r : X \to A$ and the inclusion $i : A \to X$, we have that $r \circ i = id_A$. Hence $r_* \circ i_* = id$ and r_* is a left inverse of i_*. Thus i_* is injective: We show that there is no injective map $\pi_1(S^1) \to \pi_1(R^3)$ since $\pi_1(S^1) = \mathbf{Z}$ and $\pi_1(R^3) = 0$.
   (b) (10 pts.) $X = D^2 \vee D^2$ with A equal to its boundary $S^1 \vee S^1$.
   Again $\pi_1(S^1 \vee S^1) = \mathbf{Z} * \mathbf{Z}$ and $\pi_1(D^2 \vee D^2) = 0$. Hence, there is no retractions.
   (c) (10 pts.) X a Mobius band and A its boundary circle.
   $\pi_1(A) \to \pi(X)$ is $Z \to Z$ given by $x \mapsto 2x$. Therefore any retraction r induces a map $r_* : \pi_1(X) \to \pi_1(A)$ cannot have $r_* \circ i_* = id_{\pi_1(A)}$.

2. Let $f : X \to Y$ be a homotopy equivalence.
   (a) (10 pts.) Show that f induces a bijection between the set of components of X to the set of components of Y.
   The image of a component of X under f is connected and hence goes into a component of Y. Thus f induces a map f_* from the set of components of X to the set of components of Y. Let g be the homotopy inverse. Then g induces a map g^* from the set of components of Y to that of X. $f \circ g$ induces a map from the set of components of X to the set of components of X. Since $f \circ g$ is homotopic to id_X, it follows that f_* and g_* are inverses. The reason is that the homotopy F induces a map which is the same as those induced by $f \circ g$ and id_X.
   (b) (10 pts.) Show that f induces a bijection between the set of path-components of X to the set of path-components of Y.
   Very similar to above.
   (c) (10 pts.) Show that f restricts to a homotopy equivalence between a path-component of X to a corresponding path-component of Y.
   We take the restriction $f|C : C \to D$ from a component C of X to D of Y. As an inverse we take $g|D : D \to X$. Since $g \circ f|C$ is homotopic to id_C, it follows that $g(D) \subset C$. (Omit proof). Then $g|D : D \to C$ is the homotopy inverse of $f|C$. (Omit proof.)

3.
   (a) (10 pts.) Find a nonregular 4-fold cover of a bouquet of two circles using the fundamental groups. Find also a regular one.

nonregular: diagram omitted

$$\langle a, b^2, ba^2b^{-1}, bab^2a^{-1}b^{-1}, babab^{-1}a^{-1}b^{-}\rangle$$

regular: diagram omitted

$$\langle a^2, b^2, ab^2a^{-1}baba, ba^{-1}ba\rangle$$

(b) (10 pts.) Show that a punctured torus is homotopy equivalent to the bouquet of two circles.

Remove the center of the square identified to give a torus. Then retract to the boundary of the square. The result is the wedge of two circles.

(c) (10 pts.) Find a nonregular 4-fold cover of a punctured torus using the fundamental groups. Also, find a regular one.

Simply write down the subgroup of the fundamental group of the punctured torus to get the correct answer. Otherwise, you get points off.

4. (30 pts.) Let X be the space obtained from the two-sphere S^2 and an interval by identifying the endpoints with a single point of S^2. Compute the fundamental group of X and classify the covering spaces of X up to covering equivalences.

two-sphere, I mean 2-sphere not two spheres.

X is a wedge of a circle and a 2-sphere. Thus by Van-Kampen's theorem, we obtain $\pi_1(X) = Z$ since $\pi_1(S^2) = 0$, $\pi_(S^1) = Z$ and $\pi_1(*) = 0$, and $\pi_1(X) = \pi_1(S^1) * \pi_1(S^2)/N$ where $N = 0$.

The subgroups of Z are nZ where n is a positive integer.

Hence the universal covering space is a R with 2-spheres attached at integer point. The quotients covering spaces are S^1 with n 2-spheres attached at $e^{2t\pi/n}$, $t = 0, \ldots, n-1$.

5. Let $p : \tilde{X} \to X$ be a universal covering map of X.

(a) (10 pts.) Show that X is homeomorphic to $\tilde{X}/G$ for the deck transformation group G.

Since $p : \tilde{X} \to X$ is a quotient map, there is an induced $1-1$ onto map $\tilde{X}/G \to X$ which is continuous. It is also open as we can see from elementary neighborhoods. Hence it is a homeomorphism.

(b) (10 pts.) Show that for any subgroup H of G, the induced map $\tilde{X}/H \to X$ is a covering map.

Since $H \subset G$, there is a well-defined map $\tilde{X}/H \to \tilde{X}/G$, which is continuous as well. The covering map property comes from looking at the elementary neighborhoods.

(c) (10 pts.) Show that a covering map $p_Y : Y \to X$ of X is isomorphic to $\tilde{X}/H \to X$ for some group H, and determine H from knowing p_Y.

Let K be the image subgroup of p_Y^* in $\pi_1(X, x_0)$. Then there is an anti-isomorphim $\pi_1(X, x_0) \to G$ given by sending γ to $g \in G$ such that $g(\tilde{x}_0) = \tilde{x}_0\gamma$. Let H be the image of K. Then the covering map $p_H : \tilde{X}/H \to X$ induces p_H^* with the image K. Since the covering maps are classified by subgroups, Y and $\tilde{X}/H$ are the isomorphic covering spaces.