Deformations of convex real projective structures on orbifolds

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(G, X)-structure on manifolds

Given a manifold or orbifold M, we cover it by open subsets of X pasted by elements of G. The compatibility class of the atlas of charts is a (G, X)-*structure* on M.

Orbifolds

- By an *n-dimensional orbifold* is a space modelled on finite quotients of open sets (with some compatibility conditions.)
- Let P be a convex polyhedron and we silver each side where the angles are of form π/n: *Coxeter orbifolds*.

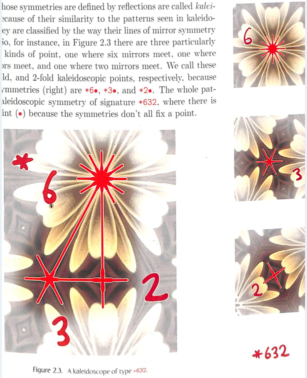
hose symmetries are defined by reflections are called *kalei-*
ecause of their similarity to the patterns seen in kaleido-
ey are classified by the way their lines of mirror symmetry
So, for instance, in Figure 2.3 there are three particularly
kinds of point, one where six mirrors meet, one where
rs meet, and one where two mirrors meet. We call these
ld, and 2-fold kaleidoscopic points, respectively, because
ymmetries (right) are *6•, *3•, and *2•. The whole pat-
aleidoscopic symmetry of signature *632, where there is
int (•) because the symmetries don't all fix a point.

Figure 2.3. A kaleidoscope of type *632.

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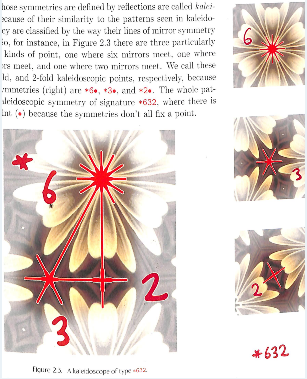
- Let P be a convex polyhedron and we silver each side where the angles are of form π/n: *Coxeter orbifolds*.
- Examples: a square with silvered edges, a triangular orbifold (Conway's picture)
- A good orbifold: M/Γ where Γ is a discrete group with a properly discontinuous action.



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Projective, affine geometry

- $\mathbb{R}P^n = P(\mathbb{R}^{n+1}) = (\mathbb{R}^{n+1} - \{O\})/\sim$ where $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{O\}$.
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$$A^n \hookrightarrow \mathbb{R}P^n, Aff(A^n) \hookrightarrow \mathrm{PGL}(n+1, \mathbb{R}).$$

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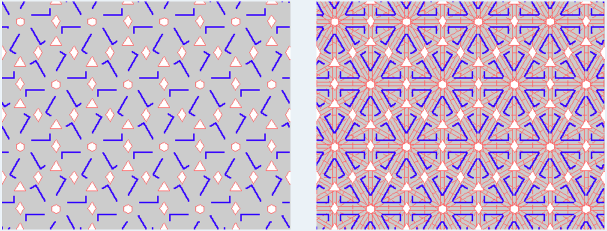

Figure: Wall-paper groups 16 and 17.

Hyperbolic geometry

- $\mathbb{R}^{1,n}$ with Lorentzian metric $q(\vec{v}) := -x_0^2 + x_1^1 + \cdots + x_n^2$.
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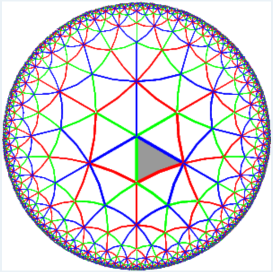

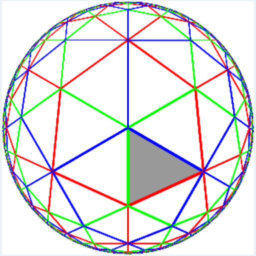

Figure: The triangle group $D^2(3, 3, 4)$ in the Poincare and Klein models by Bill Casselman.

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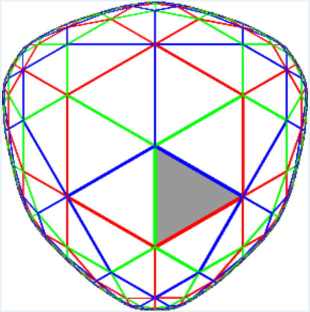
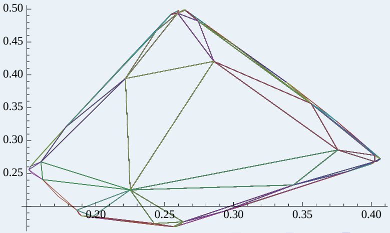
- If D is properly convex, then D/Γ is called a *properly convex real projective orbifold*.





Coxeter 3-orbifolds

We will concentrate on 3-dimensional orbifolds whose base spaces are convex polyhedra and whose sides are silvered and each edge is given an order.
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The fundamental group of the orbifold will be a Coxeter group with a presentation

$$R_i, i = 1, 2, \ldots, f, (R_i R_j)^{n_{ij}} = 1, n_{ij} \geq 2$$

where R_i is associated with silvered sides and $R_i R_j$ has order n_{ij} associated with the edge formed by the i-th and j-th side meeting.

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We keep vertices of P of form $(2, 2, n), n \geq 2, (2, 3, 3), (2, 3, 4), (2, 3, 5),$ i.e., orders of spherical triangular groups and remove others. This makes P into an open 3-dimensional orbifold with ends. (For higher-dimensional polyhedrons, we do similar operations.)

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Let $\hat{P}$ denote the differentiable orbifold with sides silvered and the edge orders realized as assigned from P with vertices removed. We say that $\hat{P}$ has a *Coxeter orbifold structure*.

Vinberg's results...

His main results is that a closed $\mathbb{R}P^n$-orbifold $\hat{P}$ is properly convex, i.e., $\hat{P}$ is a quotient of a precompact convex domain in an affine subspace of $\mathbb{R}P^n$.

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A linear reflection group is determined by the polytope given by equations $a_i \equiv 0$ for $i = 1, .., f$ and the reflection points b_i, $i = 1, .., f$. $R_i = \mathrm{I} - b_i \otimes a_i, a_i(b_i) = 2$ satisfying

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Cartan matrix: $(a_{ij} = a_i(b_j))$ satisfies

- $a_{ij} \leq 0, i \neq j$, and if $a_{ij} = 0$, then $a_{ji} = 0$.
- $a_{ii} = 2, a_{ij}a_{ji} \geq 4$, or $a_{ij}a_{ji} = 4\cos^2(\pi/n_{ij})$.
- $\mathrm{rank}(a_{ij}) = n + 1$.

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- In general, symmetric Cartan matrices can be deformed to nonsymmetric Cartan matrices $(a_{ij} = a_i(b_j))_{ij}$ and they correspond to the deformations.
- The rank of the matrix equals one $+$ the dimension of the Coxeter orbifold. The cyclic invariants $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ for distinct indices are complete invariants.
- Kac and Vinberg found first class of examples of convex $\mathbb{R}P^n$-orbifolds that are not Riemannian hyperbolic based on hyperbolic reflection triangle groups and deforming.

Deformation spaces

- The *deformation space* $\mathcal{D}(\hat{P})$ of projective structures on an orbifold $\hat{P}$ is the space of all projective structures on $\hat{P}$ quotient by isotopy group actions of $\hat{P}$.
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- A point p of $\mathcal{D}(\hat{P})$ always determines a fundamental polyhedron P up to projective automorphisms.
- We wish to understand the space where the fundamental polyhedron is always projectively equivalent to P.

This is the *restricted deformation space* of $\hat{P}$ and we denote it by $\mathcal{D}_P(\hat{P})$.

Benoist Theory of convex real projective structures on closed orbifolds

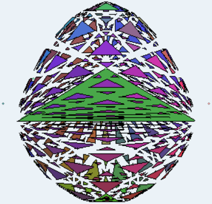
In papers "Divisibles I - IV":

Let $\mathcal{O}$ be a properly convex projective closed orbifold of dimension n.

- $\pi_1(\mathcal{O})$ is Gromov hyperbolic iff $\mathcal{O}$ is strictly convex.
- $\mathcal{O} = \Omega/\Gamma$ where $\mathrm{Cl}(\Omega) = \Omega_1 * \cdots * \Omega_m$ and Γ is a cocompact subgroup of $\mathbb{R}^{m-1} \times \mathrm{Aut}(\Omega_1) \times \cdots \times \mathrm{Aut}(\Omega_m)$.
- $\mathcal{CD}(\mathcal{O})$ is a union of components of

$$\mathrm{Hom}(\pi_1(\mathcal{O}), \mathrm{PGL}(n+1, \mathbb{R}))/\mathrm{PGL}(n+1, \mathbb{R})$$

when $\pi_1(\mathcal{O})$ has trivial virtual center and no finite index nilpotent subgroup.



Orderable Coxeter 3-orbifolds

We say that the polytope P is *orderable* if we can order the sides of P so that each side meets sides of higher index in less than or equal to 3 edges.

Definition

Let $\hat{P}$ be the orbifold obtained from P by silvering sides and removing vertices as above. We also say that the orbifold $\hat{P}$ is *orderable* if the sides of P can be ordered so that each side has no more than three edges which are either of order 2 or included in a side of higher index.

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Theorem

Let P be a convex polyhedron and $\hat{P}$ be given a normal-type Coxeter orbifold structure. Let $k(P) = \dim Aut(P)$. Suppose that $\hat{P}$ is orderable. Then $\mathcal{D}_P(\hat{P})$ is a smooth manifold of dimension $3f - e - e_2 - k(P)$ if it is not empty.

Iterated-truncation tetrahedron (ecimaedre combinatoire)

Theorem of L. Marquis

We start with a tetrahedron and cut off a vertex. We iterate this. This gives us a convex polytope P with trivalent vertices. Let $\hat{P}$ be a Coxeter 3-orbifold based on P satisfying the Andreev conditions. Then $\mathcal{D}(\hat{P})$ is diffeomorphic to $\mathbb{R}^{e_+ - 3}$.

The proof is basically very combinatorial and algebraic over $\mathbb{R}$.

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The orderbility is more general then truncation orbifold conditions; however, for compact ones, they are the same.

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- The deformation space has dimension $e - 3$ and smooth at the hyperbolic point.
- The proof involves Weil-Prasad infinitesimal rigidity.

Weakly orderble hyperbolic Coxeter orbifolds

A Coxeter 3-orbifold $\hat{P}$ is *weakly orderable* if the facets of P can be ordered so that each facet contains at most 3 edges of order 2 in a facet of higher index.

Theorem (Choi-Lee, Greene)

Let $\hat{P}$ is a closed hyperbolic Coxeter 3-orbifold. If $\hat{P}$ is weakly orderable, then at the hyperbolic structure $\mathcal{D}(\hat{P})$ is smooth and of dimension $e_+ - 3$.

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The class is quite general.

Theorem (Choi-Lee)

Let P be a simple abstract polyhedron. Suppose that P has no prismatic 3-circuit and has at most one prismatic 4-circuit. Then

$$\lim_{m\to\infty} \frac{|\{\text{weakly orderable, closed hyperbolic Coxeter 3-orbifolds } \hat{P} \text{ with edge order} \leq m\}|}{|\{\text{closed hyperbolic Coxeter 3-orbifolds } \hat{P} \text{ with edge order} \leq m\}|} = 1$$

Convex projective Dehn fillings of hyperbolic orbifolds

Let $\mathcal{O}_\infty$ be a compact Coxeter d-orbifold with a boundary $\partial\mathcal{O}_\infty$ that is a closed Coxeter $(d-1)$-orbifold admitting a Euclidean structure. A *Dehn filling* $\mathcal{O}_m$ of $\mathcal{O}_\infty$ is a Coxeter d-orbifold such that $\mathcal{O}_\infty$ is orbifold diffeomorphic to the complementary of an open neighborhood of a face r of codimension 2 of $\mathcal{O}_m$, and each interior point of r has a neighborhood modeled on $(\mathbb{R}^2/D_m) \times \mathbb{R}^{d-2}$.

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Main question

Is there a compact manifold M of dimension $d \geqslant 4$ with toral boundaries such that the interior of M admits a finite volume hyperbolic structure, and except finitely many Dehn fillings on each boundary component, each Dehn filling of M admits a properly convex real projective structure? (For Einstein metrics, Anderson Balmer answer)

Theorem

In dimension $d = 4, 5, 6$ (resp. $d = 7$), there exists a complete finite volume hyperbolic Coxeter d-orbifold $\mathcal{O}_\infty$ with holonomy representation $\rho_\infty : \pi_1(\mathcal{O}_\infty) \to \mathrm{O}_{d,1}(\mathbb{R}) \subset \mathrm{SL}^{\pm}_{d+1}(\mathbb{R})$ such that there is a sequence

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- *The image $\rho_m(W_\infty)$ is discrete and acts properly discontinuously and cocompactly (with finite covolume) on the unique properly convex domain $\Omega_m \subset \mathbb{S}^d$.*
- *The induced representation $\pi_1(\mathcal{O}_\infty)/\ker(\rho_m) \to \mathrm{SL}^{\pm}_{d+1}(\mathbb{R})$ is the holonomy representation of a properly convex real projective Dehn filling $\mathcal{O}_m$ of $\mathcal{O}_\infty$.*

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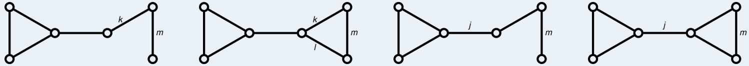
- *The representations $(\rho_m)_{m>N}$ converge algebraically to ρ_∞.*
- *The convex domains $(\Omega_m)_{m>N}$ converge to $\Omega_\infty = \mathbb{H}^d \subset \mathbb{S}^d$ in the Hausdorff topology.*

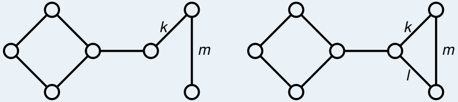
Conjecture

$\mathcal{O}_m \to \mathcal{O}_\infty$ in the Gromov-Hausdorff topology.



$k = 3, 4, 5.$ $k \geqslant l = 3, 4, 5.$ $j = 4, 5.$ $j = 4, 5.$

Table: Thirteen prime examples in dimension 4.



$k = 3, 4, 5.$ $k, l = 3, 4, 5.$ and $k \geqslant l$

Table: Nine prime examples in dimension 5

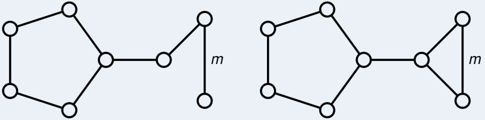


Table: Two prime examples in dimension 6

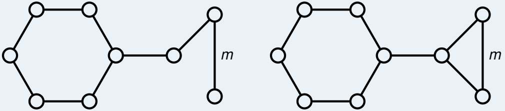


Table: Two prime examples in dimension 7

If m is finite, then the underlying polytope of G^i_m is the product of two triangles, and if $m = \infty$, then the underlying polytope of G^i_∞ is the pyramid over the prism (see Figure 4 for the Schlegel diagrams of these polytopes). We give labels on the ridges of G^i_m using the Coxeter groups W^i_m in Table 5.

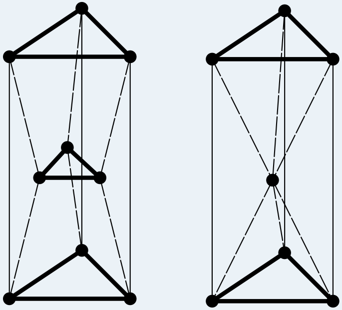


Figure: The Schlegel diagram of the product of two triangles and of a pyramid over a prism.

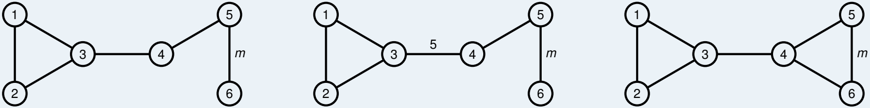


Table: W_m^1 , W_m^2 , W_m^3

We give dihedral angle between two prisms at the center triangle which appears at the missing center vertex. The proof is the computation that the angle can be deformed.

Some references

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