Linear Algebra: Midterm Exam (2005 Spring)

Justify your answers fully.

1. Prove or disprove.

   (a) (5 pts.) Q is a field.

   (b) (5 pts.) Z is a ring.

   (c) (5 pts.) $Q[x]$ is a linear algebra over Q.

   (c) (5 pts.) $Q[x]$ is a finite dimensional Z-module.

   (d) (5 pts.) Z_p for a prime p is a field.

2. Find the row echelon forms of the following matrices:

   (a) (10 pts.)

$$\begin{bmatrix} 2 & 0 & i \\ 1 & -3 & -i \\ i & 1 & 1 \end{bmatrix} \text{ where } F = C.$$

   (b) (10 pts.)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix} \text{ where } F = Q.$$

3. Let F be a fixed field. Let V be the subspace of all polynomials over F of degree less than or equal to n. Let $t_0, \dots, t_n$ be the $n+1$ distinct elements of F.

   (a) (5 pts.) Define $L_i : V \to F$ by $L_i(f) = f(t_i)$ for $i = 0, \dots, n$. Show that each L_i is a linear functional.

   (b) (5 pts.) Show that $\{L_0, \dots, L_n\}$ is a basis in V^* by showing that there exists a dual basis $\{P_0, \dots, P_n\}$ of V. Write down the fomula for every P_i.

   (c) (10 pts.) Write down the Lagrange interpolation formular for $f = x^j$ for each $j = 0, \dots, n$.

   (d) (5 pts.) Show from the above that

$$V = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^n \end{bmatrix}$$

   is invertible.

4. Let W and $W_1, \ldots, W_k$ be subspaces of a vector space V such that $\mathcal{B}_i$ is a basis of W_i for each $i = 1, \ldots, k$.

(a) (10 pts.) Show that $\dim W = \dim W_1 + \cdots + \dim W_k$ if and only if $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ is a basis of W.

(b) (10 pts.) For each $k = 1, 2, \ldots$, find an example where the above dimension equality does not hold.

5. Find the g.c.d of the each of the following pairs of polynomials.

(a) (10 pts.) $2x^5 - x^3 - 3x^2 - 6x + 4, x^4 + x^3 - x^2 - 2x - 2$.

(b) (10 pts.) $3x^4 + 8x^2 - 3, x^3 + 2x^2 + 3x + 6$.

6. (20 pts.) Let F be the field of complex numbers. We define n linear functionals on F^n $(n \geq 2)$ by

$$f_k(x_1, \ldots, x_n) = \sum_{j=1}^{n} (k - j)x_j, \quad k = 1, \ldots, n.$$

What is the dimension of the subspace annihilated by $f_1, \ldots, f_n$? Prove your claim.

7. (20 pts.) Let F be any field, and T be a linear operator on F^n. Suppose that T has n distinct eigenvalues. Prove that T is diagonalizable.