Linear Algebra: Midterm Exam (2006 Spring)

Justify your answers fully.

1. Prove or disprove.
   (a) (5 pts.) $\mathbf{Z}_{2006}$ is a field.
   (b) (5 pts.) $\mathbf{Z}_{2006}$ is a $\mathbf{Z}_2$-module.
   (c) (10 pts.) $M_{2\times 2}(\mathbf{C})$ is an $\mathbf{R}$-module.
   (d) (10 pts.) $\{A \in M_{2\times 2}(\mathbf{Z}_7) | \det A = \pm 1\}$ forms a group under matrix multiplications.
   (e) (10 pts.) $M_{2\times 2}(\mathbf{Z}_2)$ is a commutative ring with 1.

2. Let V be the vector space of all 2×2-matrix over the real field $\mathbf{R}$ and let $B \in V$.
   (a) (10 pts.) If
   $$T(A) = AB - BA,$$
   show that T is a linear transformation from V to V. Is T invertible?
   (b) (10 pts.) Express T as a matrix under a basis of V.
   (c) (10 pts.) Is the rank of T dependent on B? If so, find examples.

3. Let $\mathbf{C}^{2\times 2}$ be the complex vector space of 2×2-matrices with complex entries. Let
$$B = \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix}.$$
Let T be a linear operator on $\mathbf{C}^{2\times 2}$ defined by $T(A) = BA$.
   (a) (10pts.) Compute the rank of T.
   (b) (10pts.) Determine T^2 and its rank.

4. (20 pts.) Let V be the vector space of polynomials over the field $\mathbf{R}$ of degree ≤ 2. Let ϕ_1,ϕ_2 and ϕ_3 be the linear functionals on V defined by
$$\phi_1(f) = \int_0^1 f(t)dt, \quad \phi_2(f) = f'(1), \quad \phi_3(f) = f(0) \text{ for } f \in V.$$
Find the basis$\{f_1, f_2, f_3\}$ of V which is dual to $\{\phi_1, \phi_2, \phi_3\}$.

5. Let V be a free $\mathbf{Z}$-module and $\{v_1, \ldots, v_n\}$ be a basis.
   (a) (10pts.) Prove that there exists a dual basis of V^*.
   (b) (10pts.) Prove the the dual basis is a basis of V^*.
   (c) (10pts.) Let A be the matrix obtained by writing v_i in terms of the another basis $\{u_1, \ldots, u_n\}$. Is A in $M_{n\times n}(\mathbf{Z})$?
   (d) (10pts.) Show that $\det A = \pm 1$.