# Chapter 2: Vector spaces

Vector spaces, subspaces, basis, dimension, coordinates, row-equivalence, computations

# A vector space (V,F, +, .)

- F a field
- V a set (of objects called vectors)
- Addition of vectors (commutative, associative) $\exists 0, \forall \alpha \in V, \alpha + 0 = 0.$

$$\forall \alpha \exists ! -\alpha, \alpha + (-\alpha) = 0.$$

- Scalar multiplications $(c, \alpha) \mapsto c\alpha, c \in F, \alpha \in V$

$$1\alpha = \alpha, (c_1 c_2)\alpha = c_1(c_2\alpha), c(\alpha + \beta) = c\alpha + c\beta, (c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$$

# Examples

$$F^n = \{(x_1, \ldots, x_n) | x_i \in F\}$$

$$\begin{aligned}(x_1, \ldots, x_n) + (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n) \\ c(x_1, \ldots, x_n) &= (cx_1, \ldots, cx_n)\end{aligned}$$

– Other laws are easy to show

$$\mathbf{C}^n, (Q + \sqrt{2}Q)^n, Z_p^n$$

$$F^{mxn} = \{\{A_{ij}\} \mid A_{ij} \in F, i = 1, \ldots, m, j = 1, \ldots, n\} =$$

$$F^{mn} = \{(A_{11}, A_{12}, \ldots, A_{mn-1}, A_{mn}) \mid A_{ij} \in F\}$$

– This is just written differently

- The space of functions: A a set, F a field

$$\{f : A \to F\}, (f+g)(s) = f(s) + g(s), (cf)(s) = c(f(s))$$

  - If A is finite, this is just $F^{|A|}$. Otherwise this is infinite dimensional.

- The space of polynomial functions

$$\{f : F \to F | f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n, c_i \in F\}$$

- The following are different.

$$\begin{array}{rcl} V = \mathbf{C} = \{x + iy | x, y \in \mathbf{R}\} & , & F = \mathbf{R} \\ V = \mathbf{C} & , & F = \mathbf{C} \\ V = \mathbf{C} & , & F = \mathbf{Q} \end{array}$$

# Subspaces

- V a vector space of a field F. A subspace W of V is a subset W s.t. restricted operations of vector addition, scalar multiplication make W into a vector space.
  - +:WxW -> W, •:FxW -> W.
  - W nonempty subset of V is a vector subspace iff for each pair of vectors a,b in W, and c in F, ca+b is in W. (iff for all a,b in W, c, d in F, ca+db is in W.)
- Example:

$$\mathbf{R}^{n-1} \subset \mathbf{R}^n, \{(x_1, \ldots, x_{n-1}, 0) | x_i \in \mathbf{R}\}$$

- $S_{m\times m} = \{A \in F^{m\times m} | A^t = A\} \subset F^{m\times m}$
  is a vector subspace with field F.
- Solution spaces: Given an mxn matrix A

$$W = \{X \in F^n | AX = 0\} \subset F^n$$

$$\forall X, Y \in W c \in F, A(cX + Y) = cAX + AY = 0. \mapsto cX + Y \in W.$$

  – Example x+y+z=0 in $R^3$. x+y+z=1 (no)
- The intersection of a collection of vector subspaces is a vector subspace
- $W = \{(x, y, z) | x = 0 \text{ or} y = 0\}$ is not.

# Span(S)

$$Span(S) = \{\sum_i c_i\alpha_i | \alpha_i \in S, c_i \in F\} \subset V$$

- Theorem 3. W= Span(S) is a vector subspace and is the set of all linear combinations of vectors in S.
- Proof:

$$\alpha, \beta \in W, c \in F,$$

$$\alpha = x_1\alpha_1 + \cdots + x_m\alpha_m$$

$$\beta = y_1\beta_1 + \cdots + y_n\beta_n$$

$$c\alpha + \beta = x_1\alpha_1 + \cdots + x_m\alpha_m + y_1\beta_1 + \cdots + y_n\beta_n$$

- Sum of subsets $S_1$, $S_2$, …,$S_k$ of V

$$S_1 + S_2 + \ldots + S_k = \{\alpha_1 + \alpha_2 + \ldots + \alpha_k | \alpha_i \in S_i\}$$

- If $S_i$ are all subspaces of V, then the above is a subspace.
- Example: y=x+z subspace:

$$Span((1,1,0),(0,1,1)) = \{c(1,1,0)+d(0,1,1)|c,d \in \mathbf{R}\} = \{(c,c+d,d)|c,d \in \mathbf{R}\}$$

- Row space of A: the span of row vectors of A.
- Column space of A: the space of column vectors of A.

# Linear independence

- A subset S of V is linearly dependent if

$$\exists \alpha_1, \dots, \alpha_n \in S, c_1, \dots, c_n \in F \text{not all 0 s.t. } c_1\alpha_1 + \cdots + c_n\alpha_n = 0.$$

- A set which is not linearly dependent is called linearly independent:
  The negation of the above statement

$$\forall \alpha_1, \dots, \alpha_n \in S, \text{ there are no } c_1, \dots, c_n \in F \text{not all 0 such that } c_1\alpha_1 + \cdots + c_n\alpha_n = 0.)$$

$$\forall \alpha_1, \dots, \alpha_n \in S, \text{ if } c_1\alpha_1 + \cdots + c_n\alpha_n = 0, \text{ then } c_i = 0, i = 1, \dots, n$$

$$(1,1), (0,1), c_1(1,1) + c_2(0,1) = (c_1, c_1 + c_2) = (0,0) \mapsto c_1 = 0, c_2 = 0$$

$$c_1(1,1,1) + c_2(2,2,1) + c_3(3,3,2) = 0 \text{ for } c_1 = 1, c_2 = 1, c_3 = -1.$$

# Basis

- A basis of V is a linearly independent set of vectors in V which spans V.
- Example: $F^n$ the standard basis

$$\epsilon_1 = (1, 0, \ldots, 0), \epsilon_2 = (0, 1, \ldots, 0), \ldots, \epsilon_n = (0, 0, \ldots, 1)$$

- V is finite dimensional if there is a finite basis. Dimension of V is the number of elements of a basis. (Independent of the choice of basis.)
- A proper subspace W of V has dim W < dim V. (to be proved)

- Example: P invertible nxn matrix. $P_1,\ldots,P_n$ columns form a basis of $F^{nx1}$.
  - Independence: $x_1P_1+\ldots+x_nP_n=0$, PX=0. Thus X=0.
  - Span $F^{nx1}$: Y in $F^{nx1}$. Let $X = P^{-1}Y$. Then Y = PX. $Y= x_1P_1+\ldots+x_nP_n$.
- Solution space of AX=0. Change to RX=0.

$$\begin{array}{ccccccc} x_{k_1} & & & & + & \sum_{j=1}^{n-r} C_{1j}u_j & =0 \\ & x_{k_2} & & & + & \sum_{j=1}^{n-r} C_{2j}u_j & =0 \\ & & \ddots & & + & \vdots & =\vdots \\ & & & x_{k_r} & + & \sum_{j=1}^{n-r} C_{rj}u_j & =0 \end{array}$$

  - Basis $E_j$ $u_j=1$, other $u_k=0$ and solve above

$$x_{k_i} = -c_{ij}, \mapsto (-c_{1j}, -c_{2j}, \ldots, -c_{rj}, 0, .., 1, ..0)$$

- Thus the dimension is n-r:

- Infinite dimensional example:
- V:={f| f(x) = $c_0+c_1x+c_2x^2 + \ldots + c_nx^n$}.
  - Given any finite collection $g_1,\ldots,g_n$ there is a maximum degree k. Then any polynomial of degree larger than k can not be written as a linear combination.

- Theorem 4: V is spanned by $\beta_1, \beta_2, \ldots, \beta_m$ Then any independent set of vectors in V is finite and number is ≤ m.
  - Proof: To prove, we show every set S with more than m vectors is linearly dependent. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be elements of S with n > m.

$$\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i$$

$$\sum_{i=1}^{n} x_j\alpha_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij}\beta_i = \sum_{i=1}^{m}(\sum_{j=1}^{n} A_{ij}x_j)\beta_i$$

  - A is mxn matrix. Theorem 6, Ch 1, we can solve for $x_1, x_2, \ldots, x_n$ not all zero for

$$\sum_{j=1}^{n} A_{ij}x_j = 0, i = 1, \ldots, n$$

  - Thus

$$x_1\alpha_1 + \ldots + x_n\alpha_n = 0$$

- Corollary. V is a finite d.v.s. Any two bases have the same number of elements.
  - Proof: B,B' basis. Then |B'|≤|B| and |B|≤|B'|.
- This defines dimension.
  - dim $F^n$=n. dim $F^{mxn}$=mn.
- Lemma. S a linearly independent subset of V. Suppose that b is a vector not in the span of S. Then S∪{b} is independent.
  - Proof: $$c_1\alpha_1 + \cdots + c_m\alpha_m + kb = 0.$$
    Then k=0. Otherwise b is in the span.
    Thus, $$c_1\alpha_1 + \cdots + c_m\alpha_m = 0.$$
    and $c_i$ are all zero.

- Theorem 5. If W is a subspace of V, every linearly independent subset of W is finite and is a part of a basis of W.
- W a subspace of V. dim W ≤ dim V.
- A set of linearly independent vectors can be extended to a basis.
- A nxn-matrix. Rows (respectively columns) of A are independent iff A is invertible.

  (->) Rows of A are independent. Dim Rows A = n. Dim Rows r.r.e R of A =n. R is I -> A is inv.

  (<-) A=B.R. for r.r.e form R. B is inv. $AB^{-1}$ is inv. R is inv. R=I. Rows of R are independent. Dim Span R = n. Dim Span A = n. Rows of A are independent.

- Theorem 6.
  $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim W_1 \cap W_2$.
- Proof:
  - $W_1 \cap W_2$ has basis $a_1, \ldots, a_k$. $W_1$ has basis $a_1, .., a_k, b_1, \ldots, b_m$. $W_2$ has basis $a_1, .., a_k, c_1, \ldots, c_n$.
  - $W_1+W_2$ is spanned by $a_1, .., a_k, b_1, \ldots, b_m, c_1, \ldots, c_n$.
  - There are also independent.
    - Suppose

$$\sum_{i=1}^{l} x_i a_i + \sum_{j=1}^{m} y_j b_j + \sum_{k=1}^{n} z_k c_k = 0$$

    - Then

$$\sum_{k=1}^{n} z_k c_k = -\sum_{i=1}^{l} x_i a_i - \sum_{j=1}^{m} y_j b_j$$

$$\sum_{k=1}^{n} z_k c_k \in W_1 \text{ and } \in W_2 \qquad \sum_{k=1}^{n} z_k c_k = \sum_{i=1}^{l} d_i a_i$$

    - By independence $z_k=0$. $x_i=0, y_j=0$ also.

# Coordinates

- Given a vector in a vector space, how does one name it? Think of charting earth.
- If we are given $F^n$, this is easy? What about others?

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$$

- We use ordered basis:
  One can write any vector uniquely

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n$$

- Thus,we name

$$\alpha \mapsto (x_1, \ldots, x_n) \in F^n \qquad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\alpha]_{\mathcal{B}}$$

Coordinate (nx1)-matrix (n-tuple) of a vector.

For standard basis in $F^n$, coordinate and vector are the same.

- This sets up a one-to-one correspondence between V and $F^n$.
  - Given a vector, there is unique n-tuple of coordinates.
  - Given an n-tuple of coordinates, there is a unique vector with that coordinates.
  - These are verified by the properties of the notion of bases. (See page 50)

# Coordinate change?

- If we choose different basis, what happens to the coordinates?
- Given two bases $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}, \mathcal{B}' = \{\alpha_1', \ldots, \alpha_n'\}$
  - Write $\alpha_j' = \sum_{i=1}^{n} P_{ij}\alpha_i$

$$
\begin{aligned}
\alpha &= \sum_{j=1} x_j\alpha_j = x_1\alpha_1 + \cdots + x_n\alpha_n \\
&= \sum_{j=1}^{n} x_j'\alpha_j' = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij}\alpha_i \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} (P_{ij}x_j')\alpha_i = \sum_{i=1}^{n} (\sum_{j=1}^{n} P_{ij}x_j')\alpha_i. \\
x_i &= \sum_{j=1}^{n} P_{ij}x_j'
\end{aligned}
$$

- X=0 iff X'=0 Theorem 7,Ch1, P is invertible
- Thus, X = PX', X'=$P^{-1}$X.

$$[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}, [\alpha]_{\mathcal{B}'} = P^{-1}[\alpha]_{\mathcal{B}},$$

- Example {(1,0),(0,1)}, {(1,i), (i,1)}
  - (1,i) = (1,0)+i(0,1) (i,1) = i(1,0)+(0,1) $P = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1/2 & -i/2 \\ -i/2 & 1/2 \end{pmatrix},$
  - (a,b)=a(1,0)+b(1,0): $(a,b)_B$ =(a,b)
  - $(a,b)_{B'}$ = $P^{-1}$(a,b) = ((a-ib)/2,(-ia+b)/2).
  - We check that (a-ib)/2x(1,i)+ (-ia+b)/2x(i,1)=(a,b).