Ch. 5 Determinants

Ring

Determinant functions

Existence, Uniqueness and Properties

Rings

- A **ring** is a set K with operations
  - $(x,y) \to x+y$.
  - $(x,y) \to xy$.
  - (a) K is commutative under +
  - (b) $(xy)z=x(yz)$
  - (c) $x(y+z)=xy+xz$, $(y+z)x=yx+zx$
- If $xy=yx$, then K is a **commutative ring**.
- If there exists 1 s.t. $1x=x1=x$ for all x in K, then K is a **ring with 1**.

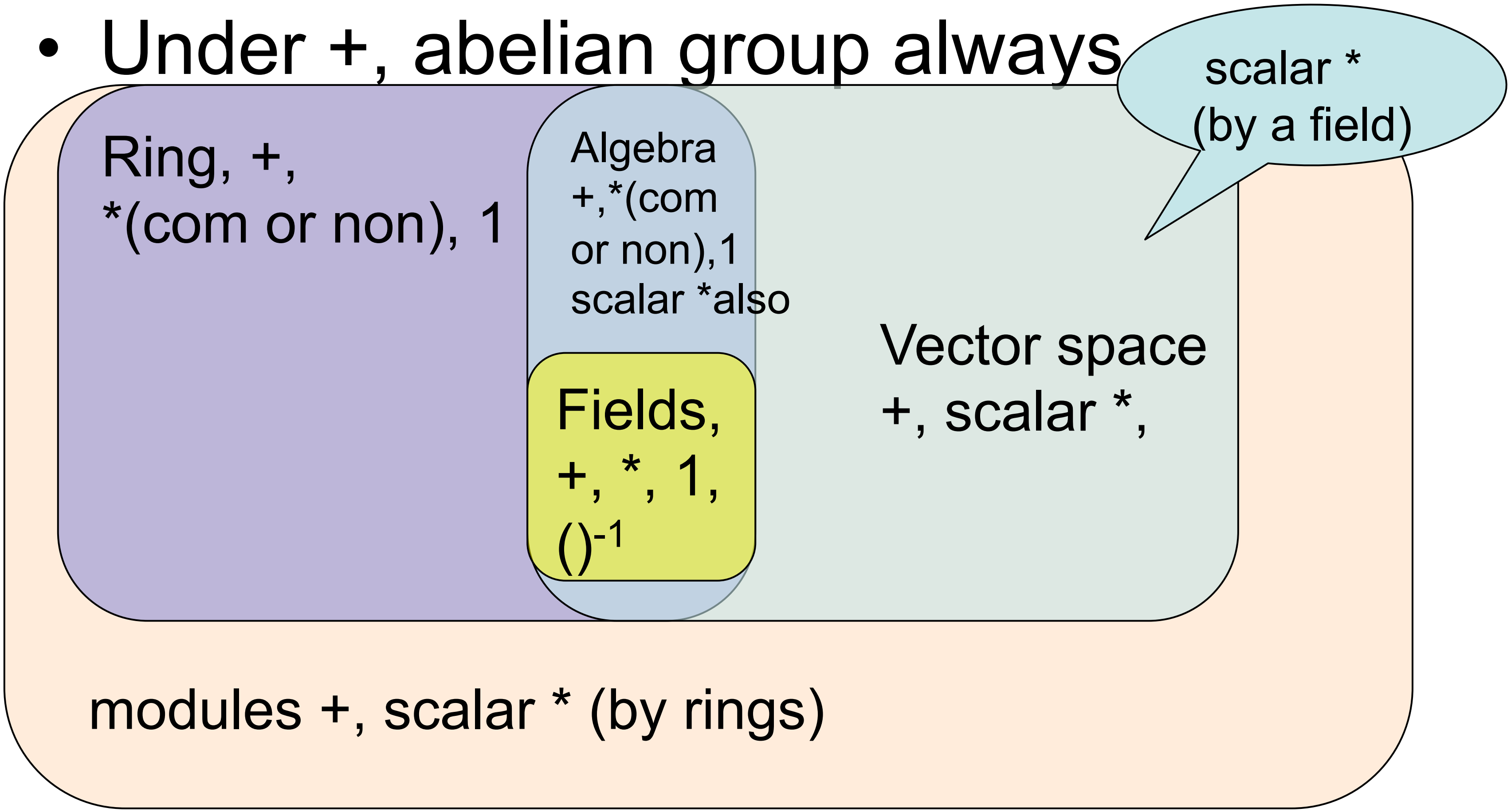
- Fields are commutative rings.
- F[x] is a commutative ring with 1.
- Z the ring of integers is a commutative ring with 1. Not a field

$$Z + iZ, Z + \sqrt{2}Z, M_{n\times n}(F), M_{n\times n}(Z), L(V)$$

- Rings with 1. Two are commutative.
- Z_n. n any positive integer is a commutative ring with 1.

Field, vector space, algebra, ring, modules…

- Under +, abelian group always



- Definition: $M_{mxn}(K)$={A_{mxn} | a_{ij} in K }, K a commutative ring with 1.
  - Sum and product is defined
  - A(B+C)=AB+AC
  - A(BC)=(AB)C.
  - m=n case: This will be a ring (not commutative in general)
- We introduce this object to prove some theorems *elegantly* in this book.

5.2. Determinant functions

Existence and Uniqueness

- K^{nxn} ={nxn matrices over K} = {n tuple of n-dim row vectors over K}
- n-linear functions

  D: K^{nxn} -> K, A -> D(A) in K.

  – D is n-linear if $D(r_1,\ldots,r_i,\ldots,r_n)$ is a linear function of r_i for each i. r_i=ith row.

$$
\begin{aligned}
D(\alpha_1, ..., c\alpha_i + \alpha_i', ..., \alpha_n) &= cD(\alpha_1, ..., \alpha_i, ..., \alpha_n) \\
&+ D(\alpha_1, ..., \alpha_i', ..., \alpha_n)
\end{aligned}
$$

- Example: $D(A) := a\, A(1,k_1)\ldots,A(n,k_n)$, $1 \le k_i \le n$, $A(i,j) := A_{ij}$.
- This is n-linear: $n=3$, $k_1=2, k_2=3$, $k_3=3$
- $D(A) = cA(1,2)A(2,3)A(3,3)$
  - $D(a_1, da_2+a_2', a_3) = ca_{12}(da_{23}+a'_{23})a_{33}$
  - $= cda_{12}a_{23}a_{33} + ca_{12}a'_{23}a_{33}$
  - $= dD(a_1,a_2,a_3) + D(a_1,a_2',a_3)$
- Proof: $D(\ldots,a_i,\ldots) = A(i,k_i)b$
  $D(\ldots.,ca_i+a'_i,\ldots) = (cA(i,k_i)+A'(i,k_i))b$
  $= cD(\ldots,a_i,\ldots)+D(\ldots,a'_i,\ldots)$.

- Lemma: A linear combination of n-linear functions is n-linear.
- Definition: D is n-linear. D is alternating if
  - (a) D(A)=0 if two rows of A are equal.
  - (b) If A' is obtained from A by interchanging two rows of A, then D(A)=-D(A').
- Definition: K a commutative ring with 1.

  D is a determinant function if D is n-linear, alternating and D(I)=1.

(The aim is to show existence and uniqueness of D)

- A 1x1 matrix D(A) = A. This is a determinant function. This is unique one.
- A 2x2 matrix. $D(A) := A_{11}A_{22} - A_{12}A_{21}$.
  - This is a determinant function
    - D(I)=1.
    - 2-linear since sum of two 2-linear functions
    - Alternating. Check (a), (b) above.
    - This is also unique:

$$
\begin{aligned}
D(A) &= D(A_{11}\epsilon_1 + A_{12}\epsilon_2, D(A_{21}\epsilon_1 + A_{22}\epsilon_2) \\
&= D(A_{11}\epsilon_1, A_{21}\epsilon_1 + A_{22}\epsilon_2) + D(A_{12}\epsilon_2, A_{21}\epsilon_1 + A_{22}\epsilon_2) \\
&= D(A_{11}\epsilon_1, A_{21}\epsilon_1) + D(A_{11}\epsilon_1, A_{22}\epsilon_2) + D(A_{12}\epsilon_2, A_{21}\epsilon_1) + D(A_{12}\epsilon_2, A_{22}\epsilon_2) \\
&= A_{11}A_{21}D(\epsilon_1, \epsilon_1) + A_{11}A_{22}D(\epsilon_1, \epsilon_2) + A_{12}A_{21}D(\epsilon_2, \epsilon_1) + A_{12}A_{22}(\epsilon_2, \epsilon_2) \\
&= A_{11}A_{22} - A_{12}A_{21}
\end{aligned}
$$

$$D(\epsilon_1, \epsilon_1) = D(\epsilon_2, \epsilon_2) = 0$$

$$D(\epsilon_2, \epsilon_1) = -D(\epsilon_1, \epsilon_2) = -D(I) = -1$$

- Lemma: D nxn n-linear over K. D(A)=0 whenever two adjacent rows are equal ➔ D is alternating.
  Proof: We show
  - D(A)=0 if any two rows of A are equal.
  - D(A')=-D(A) if two rows are interchanged.
  - (i) We show D(A')=-D(A) when two adjacent rows are interchanged.
    - $0= D(\ldots, a_i+a_{i+1}, a_i+a_{i+1},\ldots)$
      $= D(\ldots,a_i,a_i,\ldots)+ D(\ldots, a_i, a_{i+1},\ldots)$
      $+ D(\ldots, a_{i+1} , a_i ,\ldots) + D(\ldots, a_{i+1} , a_{i+1} ,\ldots)$
      $= D(\ldots, a_i, a_{i+1},\ldots) + D(\ldots, a_{i+1} , a_i ,\ldots)$

– (ii) Say B is obtained from A by interchanging row i with row j. i<j.

$$\alpha_1, \alpha_2, ..., \alpha_i, \underbrace{\alpha_{i+1}, ..., \alpha_j}_{j-i}, ..., \alpha_n$$

$$\alpha_1, \alpha_2, ..., \alpha_{i-1}, \underbrace{\alpha_{i+1}, ..., \alpha_j}_{j-i}, \alpha_i ..., \alpha_n$$

$$\alpha_1, \alpha_2, ..., \alpha_{i-1}, \alpha_j, \underbrace{\alpha_{i+1}, ..., \alpha_{j-1}}_{j-i-1}, \alpha_i ..., \alpha_n$$

- $D(B)= (-1)^{2(j-i)-1} D(A)$, $D(B)=-D(A)$.

– (iii) D(A)=0 if A has two same i, j rows: Let B be obtained from A so that has same adjacent rows. Then D(B)=-D(A), D(A)=0.

- Construction of determinant functions:
  - We will construct the functions by induction on dimensions.
- Definition: $n>1$. A nxn matrix over K. $A(i|j)$ $(n-1)\times(n-1)$ matrix obtained by deleting ith row and jth column.
- If D is $(n-1)$-linear, A nxn, define $D_{ij}(A) = D[A(i|j)]$.
- Fix j. Define

$$E_j(A) := \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$$

- Theorem 1: $n>1$.
  - E_j is an alternating n-linear function.
  - If D is a determinant, then E_j is one for each j.
- This constructs a determinant function for each n by induction.
- Proof: $D_{ij}(A)$ is linear of any row except the ith row.
  - $A_{ij} D_{ij}(A)$ is n-linear
  - E_j is n-linear

– We show $E_j(A)=0$ if A has two equal adjacent rows.

- Say $a_k=a_{k+1}$. $D[A(i|j)] = 0$ if $i \neq k, k+1$.

$$E_j(A) = (-1)^{k+j} A_{kj} D_{kj}(A) + (-1)^{k+1+j} A_{(k+1)j} D_{(k+1)j}(A)$$

$$\alpha_k = \alpha_{k+1}, A_{kj} = A_{(k+1)j}, D_{kj}(A) = D_{(k+1)j}(A)$$

- Thus $E_j(A)=0$. E_j is alternating n-linear function.

– If D is a determinant, then so is E_j.

$$E_J(I_{n\times n}) = \sum_{i=1}^{n} (-1)^{i+j} I_{ij} D_{ij}(I) = (-1)^{2j} \delta_{jj} D_{jj}(I) = D(I_{(n-1)\times(n-1)}) = 1$$

- Corollary: K commutative ring with 1. There exists at least one determinant function on K^{nxn}.
- Proof: K^{1x1}, K^{2x2} exists
  $K^{n-1xn-1}$ exists -> K^{nxn} exists by Theorem 1.

Uniqueness of determinant functions

- Symmetric group Sn ={f:{1,2,…,n} -> {1,2,…,n}|f one-to-one, onto}
- Facts: Any f can be written as a product of interchanges (i,j):
  - Given f, the product may be many.
  - But the number is either even or odd depending only on f.
- Definition: sgn(f) = 1 if f is even, =-1 if f is odd.

- Claim: D a determinant

$$D(\epsilon_{\sigma 1}, ..., \epsilon_{\sigma n}) = \pm D(\epsilon_1, ..., \epsilon_n) = \pm 1$$

$$D(\epsilon_{\sigma 1}, ..., \epsilon_{\sigma n}) = \mathrm{sgn}\sigma$$

- Proof: $D(\epsilon_{\sigma 1}, ..., \epsilon_{\sigma n})$ is obtained from I by applying $\iota_1, \cdots, \iota_m$ to D(I).
  - Each application changes the sign of the value once.
- Consequence: sgn is well-defined.

- We show the uniqueness of the determinant function by computing its formula.
- Let D be alternating n-linear function.
- A a nxn-matrix with rows $a_1, \ldots, a_n$.
- $e_1, \ldots, e_n$ rows of I.

$$\alpha_i = \sum_{j=1}^{n} A(i,j)\epsilon_j$$

$$
\begin{aligned}
D(A) &= D(\textstyle\sum_j A(1,j)\epsilon_j, \alpha_2, ..., \alpha_n) \\
&= \textstyle\sum_{k_1} A(1,k_1) D(\epsilon_{k_1}, \alpha_2, ..., \alpha_n)
\end{aligned}
$$

$$\alpha_2 = \sum_{k_2} A(2,k_2)\epsilon_{k_2}$$

$$
\begin{aligned}
D(\epsilon_{k_1}, \alpha_2, ..., \alpha_n) &= D(\epsilon_{k_1}, \textstyle\sum_{k_2} A(2,k_2)\epsilon_{k_2}, ..., \alpha_n) \\
&= \textstyle\sum_{k_2} A(2,k_2) D(\epsilon_{k_1}, \epsilon_{k_2}, ..., \alpha_n) \\
D(A) &= \textstyle\sum_{k_1} A(1,k_1) \sum_{k_2} A(2,k_2) D(\epsilon_{k_1}, \epsilon_{k_2}, ..., \alpha_n)
\end{aligned}
$$

- By induction, we obtain

$$D(A) = \textstyle\sum_{k_1, k_2, ..., k_n} A(1,k_1)A(2,k_2)\cdots A(n,k_n) D(\epsilon_{k_1}, \epsilon_{k_2}, ..., \epsilon_{k_n})$$

$$D(\epsilon_{k_1}, ..., \epsilon_{k_n}) = 0$$

- if $\{k_1, ..., k_n\}$ is not distinct.
- Thus {1,…,n}->{k_1,…,k_n} is a permutation.

$$D(A) = \sum_{\sigma \in S_n} A(1, \sigma 1)A(2, \sigma 2) \cdots A(n, \sigma n)D(\epsilon_{\sigma 1}, ..., \epsilon_{\sigma n})$$

$$D(A) = \sum_{\sigma \in S_n} \text{sgn}\sigma A(1, \sigma 1)A(2, \sigma 2) \cdots A(n, \sigma n)D(I)$$

$$det(A) = \sum_{\sigma \in S_n} \text{sgn}\sigma A(1, \sigma 1)A(2, \sigma 2) \cdots A(n, \sigma n)$$

- Theorem 2: D(A) = det(A)D(I) for D alternating n-linear.
  - Proof: proved above.
- Theorem 3: det(AB)=(detA)(det B).
- Proof: A, B nxn matrix over K.
  - Define D(A)=det(AB) for B fixed.
  - $D(a_1,\ldots,a_n) = \det(a_1B,\ldots,a_nB)$.
  - D is n-linear as a -> aB is linear.
  - D is alternating since if $a_i=a_{i+1}$, then D(A)=0.

- $D(A) = \det A \, D(I)$.
- $D(I) = \det(IB) = \det B$.
- $\det AB = D(A) = \det A \det B$.

- Fact: $sgn: S_n \to \{-1,1\}$ is a homomorphism. That is, $sgn(\sigma\tau)=sgn(\sigma)sgn(\tau)$.
- Proof: $\sigma = \sigma_1 \ldots \sigma_n$, $\tau=\tau_1 \ldots \tau_m$: interchanges. $\sigma\tau = \sigma_1 \ldots \sigma_n \tau_1 \ldots \tau_m$.
- Another proof:

$$sgn(\sigma\tau) = \det(\sigma\tau(I)) = \det(\sigma(I)\tau(I))$$

$$= \det(\sigma(I))\det(\tau(I)) = sgn(\sigma)sgn(\tau)$$