6.3. Annihilating polynomials

Cayley-Hamilton theorem

Polynomials and transformations

- (p+q)(T) = p(T)+q(T)
- (pq)(T)=p(T)•q(T)
- Ann(T) = {p in F[x]| p(T)=0} is an ideal.
- Proof:
  - p,q in Ann(T) k in F -> p+kq(T)=0-> p+kq in Ann(T).
  - p in Ann(T), q in F[x] -> pq(T)=p(T)q(T)=0 -> pq in Ann(T).
- Ann(T) is strictly bigger than {0}.

- Proof: We show that there exists a nonzero polynomial f in Ann(T) for any T:V^n->V^n.
- $I, T, T^2, \ldots, T^{n^2}$:1+n^2 operators in L(V,V).
- dimL(V,V)= n^2. Therefore, there exists a relation

$$c_0 I + c_1 T + c_2 T^2 + \ldots + c_{n^2} T^{n^2} = 0$$

- Every polynomial ideal is of form fF[x].

- Definition: T:V->V. V over F. (finite dim.)

The minimal polynomial of T is the unique monic generator of Ann(T).

- How to obtain the m.poly?
- Proposition: the m.poly p is characterized by
  1. P is monic.
  2. P(T)=0
  3. No polynomial f s.t. f(T)=0 and has smaller degree than p.

- A similar operators have the same minimal polynomials: This follows from:
- $f(GTG^{-1})=0 <-> f(T)=0$.
- Proof:
  - $(GTG^{-1})^i = GTG^{-1}GTG^{-1}\dots GTG^{-1}=GT^iG^{-1}$
  - $0=c_0I+c_1 GTG^{-1} + c_2 (GTG^{-1})^2 + \dots + c_n(GTG^{-1})^n <->$
  - $0=G(c_0I+c_1 T + c_2 T^2 + \dots + c_nT^n)G^{-1}$

- Theorem 3. T a linear operator on V^n. (A nxn matrix). The characteristic and minimal polynomial of T(of A) have the same roots, except for multiplicities.
- Proof: p a minimal polynomial of T
  - We show that
  - p(c)=0 <-> c is a characteristic value of T.
    - (->) p(c)=0. p=(x-c)q.
    - deg q < deg p. q(T)≠0 since p has minimal degree
    - Choose b s.t. q(T)b ≠0. Let a=q(T)b.
    - 0=p(T)b = (T-cI)q(T)b = (T-cI)a
    - c is a characteristic value.

- (<-) $Ta=ca$. $a\neq 0$.
- $p(T)a=p(c)a$ by a lemma.
- $a\neq 0$, $p(T)=0$ -> $p(c)=0$.

- T diagonalizable. (We can compute the m.poly)
  - $c_1,\ldots,c_k$ distinct char. values.
  - Then $p=(x-c_1)\ldots(x-c_k)$.
- Proof: If *a* is a char. Vector, then one of $T-c_1I,\ldots,T-c_kI$ sends a to 0.
  - $(T-c_1I)\ldots(T-c_kI)a=0$ for all char. v. *a.*
  - Characteristic vectors form a basis a_i.
  - $(T-c_1I)\ldots(T-c_kI)=0$.
  - $p=(x-c_1)\ldots(x-c_k)$ is in Ann(T).
  - p has to be minimal since p has to have all these factors by Theorem 3.

- Caley-Hamilton theorem: T a linear operator V. If f is a char. poly. for T, then f(T)=0. i.e., f in Ann(T). The min. poly. p divides f.
- Proof: highly abstract:
  - K a commutative ring of poly in T.
  - $\{a_1,\ldots,a_n\}$ basis for V.
  - $$Ta_i = \sum_{j=1}^{n} A_{ji} a_j \quad \text{->} \quad \sum_{j=1}^{n} (\delta_{ij} T - A_{ji} I) a_j = 0$$

    for i=1,…,n

– Let B be a matrix in K^{nxn} with entries:

$B_{ij} = \delta_{ij} T - A_{ji} I$

- e.g. n=2.

$$B = \begin{bmatrix} T - A_{11}I & -A_{21}I \\ -A_{12}I & T - A_{221}I \end{bmatrix}$$

- det B = $(T-A_{11}I)(T-A_{22}I)-A_{12}A_{21}I = f_T(T)$.

– For all n, det B = $f_T(T)$. f_T char. poly.(omit proof)

– We show f_T (T)=0 or equiv. f(T)a_k=0 for each k.

– (6-6) $\sum_{j=1}^{n} B_{ij} a_j = 0$ by (*).

– Let B' = adj B.

$$\sum_{j=1}^{n} B'_{ki} B_{ij} a_j = 0. \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} B'_{ki} B_{ij} a_j = 0.$$

$$\sum_{j=1}^{n} (\sum_{i=1}^{n} B'_{ki} B_{ij}) a_j = 0. \qquad \sum_{j=1}^{n} (\delta_{kj} \det B) a_j = 0.$$

- $\det B a_k = 0.$ for each k.
- det B=0. $f_T(T)=0$.

- Characteristic polynomial gives informations on the factors of the minimal polynomials.