6.4. Invariant subspaces

Decomposing linear maps into smaller pieces.

Later-> Direct sum decomposition

- T:V->V. W in V a subspace.
- W is invariant under T if T(W) in W.
- Range(T), null(T) are invariant:
  - T(range T) in range T
  - T(null T)=0 in null T
- Example: T, U in L(V,V) s.t. TU=UT.
  - Then range U and null U are T-invariant.
  - a=Ub. Ta=TU(b) = U(Tb).
  - Ua=0. UT(a)=TU(a)=T(0)=0.
- Example: Differential operator on polynomials of degree ≤n.

- When W in V is inv under T, we define T_W: W -> W by restriction.
- Choose basis $\{a_1, \ldots, a_n\}$ of V s.t. $\{a_1, \ldots, a_r\}$ is a basis of W.
- Then

$$Ta_j = \sum_{i=1}^{r} A_{ij} a_i, \; j = 1, \ldots, r$$

$$A_{ij} = 0, \; j = 1, \ldots, r, \; i = r+1, \ldots, n$$

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

– B rxr, C rx(n-r), D (n-r)x(n-r)

- Conversely, if there is a basis, where A is above block form, then there is an invariant subspace corr to $a_1,\ldots,a_r$.

Lemma: W invariant subspace of T.

- Char.poly of T_W divides char poly of T.
- Min.poly of T_W divides min.poly of T.

- Proof: det(xI-A)= det(xI-B)det(xI-C).
  - f(A) = c_0I+ c_1A^2+….+A^n.

$$A^k = \begin{bmatrix} B^k & C_k \\ 0 & D^k \end{bmatrix}$$

$$f(A) = \begin{bmatrix} f(B) & C^*{}_k \\ 0 & f(D) \end{bmatrix}$$

- f(A)=0 -> f(B)=0 also.
- Ann(A) is in Ann(B).
- Min.poly B divides min.poly.A. by the ideal theory.

- Example 10: W subspace of V spanned by characteristic vectors of T.
  - $c_1,\ldots,c_k$ char. values of T (all).
  - W_i char. subspace associated with c_i. B_i basis.
  - $B'=\{B_1,\ldots,B_k\}$ basis of W. $B'=\{a_1,..,a_r\}$
  - $\dim W = \dim W_1 + \ldots + \dim W_k$
  - $Ta_i = t_i a_i$. $i=1,\ldots,r$

$$a = x_1 a_1 + \ldots + x_r a_r \qquad Ta = t_1 x_1 a_1 + \ldots + t_r x_r a_r$$

  - Thus, W is invariant under T.

– The characteristic polynomial of T_w is

$$g = (x - c_1)^{e_1} \ldots\ldots (x - c_k)^{e_k}$$

– where e_i = dim W_i

- Recall:
- Theorem 2. T is diag <-> $e_1 + \ldots + e_k = n$.
- Consider restrictions of T to sums $W_1 + \ldots + W_j$ for any j. Compare the characteristic and minimal polynomials.

T-conductors

- We introduce T-conductors to understand invariant subspaces better.
- Definition: W is invariant subspace of T. T-conductor of a in V $=S_T(a;W)=\{g \text{ in } F[x] \mid g(T)a \text{ in } W\}$
- If W={0}, then $S_T(a;\{0\})$ = T-annihilator of a. (not nec. equal to Ann(T)).

- Example: V = R^4. W=R^2. T given by a matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

- Then S((1,0,0,0);W)?
- $c(1,0,0,0)+dT(1,0,0,0)+eT^2(1,0,0,0)+\ldots$
- Easy to see c=d=0.
- Equals $x^2F[x]$

- Lemma. W is invariant under T -> W is invariant under f(T) for any f in F[x]. S(a;W) is an ideal.
- Proof:
  - b in W, T b in W,…, T^k b in W. f(b) in W.
  - S(a;W) is a subspace of F[x].
    - (cf+g)(T)(a) = (cf(T)+g(T))a = cf(T)a+g(T)a in W if f,g in S(a;W).
  - S(a;W) is an ideal in F[x].
    - f in F[x], g in S(a;W). Then fg(T)(a)=f(T)g(T)(a) =f(T)(g(T)(a)) in W. fg in S(a;W).

- The unique monic generator of the ideal S(a;W) is called the T-conductor of a into W. (T-annihilator if W={0}).
- S(a;W) contains the minimal polynomial of T (p(T)a=0 is in W).
- Thus, every T conductor divides the minimal polynomial of T. This gives a lot of information about the conductor.

- Example: Let T be a diagonalizable transformation. $W_1,\ldots,W_k$.
  - W_i =null($T-c_iI$).
  - $(x-c_i)$ is the conductor of any nonzero-vector a into

$$W_1 + \ldots + W_{i-1} + W_{i+1} + \ldots + W_n$$

  - Needed condition: a is a sum of vectors in W_js with nonzero W_i vector.

Application

- T is triangulable if there exists an ordered basis s.t. T is represented by a triangular matrix.
- We wish to find out when a transformation is triangulable.

- Lemma: T in L(V,V). V n-dim v.s.over F. min.poly T is a product of linear factors.

$$p = (x - c_1)^{r_1} \ldots .(x - c_k)^{r_k}, c_i \in F$$

  Let W be a proper invariant suspace for T. Then there exists a in V s.t.
  - (a) a not in W
  - (b) (T-cI)a in W for some char. value of T.
- Proof: Let b in V. b not in W.
  - Let g be T-conductor of b into W.
  - g divides p.

$$g = (x - c_1)^{e_1} \ldots.(x - c_k)^{e_k}, 0 \le e_i \le r_i$$

- Some $(x-c_j)$ divides g.
- $g=(x-c_j)h$.
- Let a=h(T)b is not in W since g is the minimal degree poly sending b into W.
- $(T-c_j)a = (T-c_j)h(T)b = g(T)b$ in W.
- We obtained the desired a.

- Theorem 5. V f.d.v.s. over F. T in L(V,V). T is triangulable <-> The minimal polynomial of T is a product of linear polynomials over F.
- Proof: (<-) $p=(x-c_1)^{r\_1}\ldots(x-c_k)^{r\_k}$.
  - Let W={0} to begin. Apply above lemma.
  - There exists $a_1 \neq 0$, $(T-c_iI)a_1 = 0$. $Ta_1=c_ia_1$.
  - Let $W_1=\langle a_1 \rangle$.
  - There exists $a_2\neq 0$, $(T-c_jI)a_2$ in W_1. $Ta_2= c_ja_2+a_1$
  - Let $W_2=\langle a_1,a_2 \rangle$. So on.

- We obtain a sequence $a_1, a_2, \ldots, a_i, \ldots$
- Let $W_i = < a_1, a_2, \ldots, a_i >$.
- a_{i+1} not in W_i s.t. $(T - c_{j\_i+1} I) a_{i+1}$ in $W_{i.}$
- $T a_{i+1} = c_{j\_i+1}\, a_{i+1}$ + terms up to a_i only.
- Then $\{a_1, a_2, \ldots, a_n\}$ is linearly independent.
  - a_{i+1} cannot be written as a linear sum of a_1, a_2, …,a_i by above. -> independence proved by induction.
- Each subspace $<a_1, a_2, \ldots, a_i>$ is invariant under T.
  - Ta_i is written in terms of $a_1, a_2, \ldots, a_i$.

– Let the basis B= {a_1, a_2,…,a_n}. Then

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

- (->) T is triangulable. Then xI-$[T]_B$ is again triangular matrix.
  Char T =f= $(x-c_1)^{d\_1}\dots(x-c_k)^{d\_k}$.
  - $(T-c_1I)^{d\_1}\dots(T-c_kI)^{d\_k}$ (a_i) =0 by direct computations.
  - f is in Ann(T) and p divides f
  - p is of the desired form.

- Corollary. F algebraically closed. Every T in L(V,V) is triangulable.
- Proof: Every polynomial factors into linear ones.
- F=C complex numbers. This is true.
- Every field is a subfield of an algebraically closed field.
- Thus, if one extends fields, then every matrix is triangulable.

Another proof of Cayley-Hamilton theorem:

- Let f be the char poly of T.
- F in F’ alg closed.
- Min.poly T factors into linear polynomials.
- T is triangulable over F’.
- Char T is a prod. Of linear polynomials and divisible by p by Theorem 5.
- Thus, Char T is divisible by p over F also.

- Theorem 6. T is diagonalizable <-> minimal poly $p=(x-c_1)\ldots(x-c_k)$. ($c_1,\ldots, c_k$ distinct).
- Proof: -> p.193 done already
  - (<-) Let W be the subspace of V spanned by all char.vectors of T.
  - We claim that W=V.
  - Suppose $W \neq V$.
    - By Lemma, there exists a not in W s.t.
    - $b = (T-c_jI)a$ is in W.
    - $b = b_1+\ldots+b_k$ where $Tb_i= c_ib_i$. $i=1,\ldots,k$.
    - $h(T)b =h(c_1)b_1+\ldots+h(c_k)b_k$ for every poly. h.
    - $p=(x-c_j)q$. $q(x)-q(c_j)=(x-c_j)h$.

- $q(T)a - q(c_j)a = h(T)(T - c_jI)a = h(T)b$ in W.
- $0 = p(T)a = (T - c_jI)q(T)a$
- $q(T)a$ in W.
- $q(c_j)a$ in W but a not in W.
- Therefore, $q(c_j) = 0$.
- This contradicts that p has roots of multiplicities ones only.
- Thus W=V and T is diagonalizable.