6.6 Direct sum decompositions

The relation to projections

- $W_1,\ldots,W_k$ in V subspaces. $W_1,\ldots,W_k$ are linearly independent if $a_1+\ldots+a_k=0$ implies $a_i=0$ for each i.
- Lemma. TFAE:
  - $W_1,\ldots,W_k$ are independent.
  - $W_j \cap (W_1 + \ldots + W_{j-1}) = \{0\}, j = 2,\ldots,k.$
  - Bi basis for W_i -> $B=\{B_1,\ldots,B_k\}$ is a basis for W.
- Proof: omit
- We write $$W = W_1 \oplus \cdots \oplus W_k$$

- Example:

$$R^3 = R^1 \oplus R^2$$

- Example: $V = M^{nxn}(F)$
  - W_1 all symmetric matrices: $A^t=A$.
  - W_2 all antisymmetric matrices: $A^t=-A$.
  - Then $V=W_1 \oplus W_2$.
    - $A=A_1+A_2$, $A_1=(A+A^t)/2$, $A_2=(A-A^t)/2$.
- Projections: $E:V\to V$, $E^2=E$. E is called a projection.

- Example: E: (x,y,z)->(x,y,0).
- Properties: R=range of E. N=null E.
  - b in R <-> Eb=b.
    - (->) b=Ea, Eb=E(Ea)=Ea=b.
    - (<-) b=Eb. Done
  - I-E is a projection also:
    - (I-E)(I-E)=I-2E+E^2 = 1-2E+E=1-E.
  - Im(I-E)= null E:
    - Let a=(I-E)b. a=b-Eb. Ea=Eb-E^2b=Eb-Eb=0. a in null E.
    - a in null E. Ea=0. (I-E)a=a-Ea=a. a in Im(I-E).

- a in V. a=Ea+(a-Ea)=Ea+(I-E)a.
- V=Im E ⊕ Im(I-E).
  - Ea=(I-E)b. Ea=E^2a=Eb-E^2b=Eb-Eb=0. (I-E)b=0 also.
- V=R ⊕N.
  - Im E=R. Im(I-E)=N

- Projection is diagonalizable:
  - Let {a_1,…,a_r} be a basis of R.
  - {a_{r+1},…, a_n} a basis for N.
  - B ={a_1,…,a_n} a basis for V which diagonalizes E.

$$[E]_B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

- Cases where there are a number of projections (commuting)
- Theorem 9. If $V= W_1\oplus\ldots \oplus W_k$, then there exists k linear operators $E_1,\ldots,E_k$ on V s.t.
  - (i) Each E_i is a projection. $E_i^2=E_i$.
  - (ii) $E_iE_j=0$ if $i\neq j$. (commuting)
  - (iii) $I=E_1+\ldots+E_k$.
  - (iv) Range $E_i = W_i$.
- Conversely, if $E_1,..,E_k$ are k linear operators satisfying (i)-(iii), then for W_i:=range E_i, $V= W_1\oplus\ldots \oplus W_k$.

- Proof:
  - (->) $V = W_1 \oplus \ldots \oplus W_k$.
    - $a = a_1 + \ldots + a_k$, a_i in W_i. Uniquely written.
    - Define $E_j a = a_j$.
    - Then $E_j^2 = E_j$.
    - $N(E_j) = W_1 \oplus \ldots \oplus W_{j-1} \oplus W_{j+1} \oplus \ldots \oplus W_k$.
    - $a = E_1 a + \ldots + E_k a$. $I = E_1 + \ldots + E_k$.
    - $E_i E_j = 0$ if $i \neq j$. (W_j is in $N(E_i)$).
  - (<-) Let $E_1, \ldots, E_k$ linear operators.
    - $a = E_1 a + \ldots + E_k a$ by (iii).
    - $V = W_1 + \ldots + W_k$.
    - The expression is unique.

- $a = a_1+\ldots+a_k$. a_i in $W_i = \mathrm{Im}\ E_i$. $a_i = E_i b_i$.
- $E_j a = E_j(a_1+\ldots+a_k)= E_j a_j = E_j E_j b_j = E_j b_j= a_j$.
- By (b) of the Lemma, $W_1,\ldots,W_k$ are independent.
- $V= W_1 \oplus \ldots \oplus W_k$

- Note: A finite sum of any collection of distinct E_i is a projection.
  - Check:

$$(E_{i_1} + ... + E_{i_k})^2 = E_{i_1} + ... + E_{i_k}$$

6.7. Invariant direct sums

- $V = W_1 \oplus \ldots \oplus W_k$, W_i T-invariant.
  - $B=\{B_1,\ldots,B_k\}$

$$[T]_B = A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

  - Block form $$A_i = [T]_{B_i}$$

- Theorem 10: $T:V\to V$. $V= W_1\oplus\ldots\oplus W_k$ respective $E_1,\ldots,E_k$.
  - Each W_i is T-invariant <-> $TE_i=E_iT$, $i=1,\ldots,k$.
- Proof:(<-) Let a in W_j. $a=E_ja$.
  - $Ta=T(E_ja)= E_jT(a)$. Ta in W_j.
  - W_j is T-invariant.
  - (->) $a=E_1a+\ldots+E_ka$.
  - $Ta=T E_1a+\ldots+T E_ka$.
  - Since $T(E_ja)$ in W_j , $T(E_ja)=E_jb_j$ for some b_j.

- $E_jTa = E_jTE_1a + \ldots + E_jTE_ka = E_jT\, E_ja = TE_ja$. Thus, $E_jT = TE_j$

- Theorem 11. T in L(V,V). T is diagonalizable. $c_1,\ldots,c_k$ distinct char. Value of T. Then there exists projections $E_1,\ldots,E_k$ s.t.
  - (i) $T = c_1E_1 + \ldots + c_kE_k$
  - (ii) $I = E_1 + \ldots + E_k$ (iii) $E_iE_j = 0$ $i \neq j$. (iv) $E_i^2 = E_i$.
  - (v) Range E_i=char.v.s. of T ass. c_i.
  - Conversely, given distinct $c_1,\ldots,c_k$, $E_1,\ldots,E_k$ with (i)-(iii). Then T is diagonalizable with char. values $c_1,\ldots,c_k$ and (iv)(v) also hold.

- Proof: (->) T diagonalizable. $c_1,\ldots,c_k$ dist. Char. Values.
  - $V = W_1 \oplus \ldots \oplus W_k$, W_i associated with c_i.
  - $a=E_1a+\ldots+E_ka$.
  - $Ta=T E_1a+\ldots+T E_ka=c_1 E_1a+\ldots+c_kE_ka$.
  - $T= c_1 E_1+\ldots+c_kE_k$.
  - (<-) We need to prove (iv)(v) and T is diagonalizable.
  - (iv) $E_iI = E_i(E_1+\ldots+E_k)=E_i^2$.
  - $T= c_1 E_1+\ldots+c_kE_k$ by(i).
  - $T E_i = c_iE_i$ by (iii).
  - Since E_i is not zero, c_i is a char.value.
  - $T-cI= (c_1-c)E_1+\ldots+(c_k-c)E_k$.

- If $c \neq c_i$ for all I, then T-cI has the null-space {0}.
- T has char.values $c_1,\dots,c_k$.
- T is diagonalizable since char.v.s. span V.
- (v) null($T-c_iI$)=Im E_i.
  - ($\supset$) $(T-c_iI)E_ia= ((c_1-c_i)E_1+\dots+(c_{i-1}-c_i)E_{i-1}+ (c_{i+1}-c_i)E_{i+1}+(c_k-c_i)E_k)E_ia = 0$ by (iii).
  - ($\subset$) If $Ta=c_ia$, then $(c_1-c_i)E_1a+\dots+(c_{i-1}-c_i)E_{i-1}a+ (c_{i+1}-c_i)E_{i+1}a+(c_k-c_i)E_ka =0$.
  - $(c_j-c_i)E_ja=0$ for all $j \neq i$.
  - $E_ja=0$ for all $j \neq i$.
  - a in Im E_i.

- Remark: diagonalizable operator T is uniquely determined by $c_1,\ldots,c_k$ and $E_1,\ldots,E_k$.
- $T= c_1 E_1+\ldots+c_kE_k.$
  - $g(T)= g(c_1) E_1+\ldots+g(c_k)E_k$
  - Proof omitted.
- Let

$$p_j = \prod_{i\neq j}\frac{(x-c_i)}{(c_j-c_i)} \qquad p_j(T) = \sum_{i=1}^{k} p_j(c_i)E_i = \sum_{i=1}^{k}\delta_{ij}E_i = E_j$$

  - Thus, E_j is a polynomial of T and hence commutes with T.