7.2. Cyclic decomposition and rational forms

Cyclic decomposition

Generalized Cayley-Hamilton

Rational forms

- We prove existence of vectors $a_1,..,a_r$ s.t. $V=Z(a_1;T)\oplus\ldots\oplus Z(a_r;T)$.
- If there is a cyclic vector a, then $V=Z(a;T)$. We are done.
- Definition: T a linear operator on V. W subspace of V. W is T-admissible if
  - (i) W is T-invariant.
  - (ii) If f(T)b in W, then there exists c in W s.t. f(T)b=f(T)c.

- Proposition: If W is T-invariant and has a complementary T-invariant subspace, then W is T-admissible.
- Proof: $V = W \oplus W'$. T(W) in W. T(W') in W'. b=c+c', c in W, c' in W'.
  - f(T)b=f(T)c+f(T)c'.
  - If f(T)b is in W, then f(T)c'=0 and f(T)c is in W.
  - f(T)b=f(T)c for c in W.

- To prove $V=Z(a_1;T)\oplus \ldots \oplus Z(a_r;T)$, we use induction:
- Suppose we have $W_j=Z(a_1;T)+\ldots+Z(a_j;T)$ in V.
  - Find a_{j+1} s.t. $W_j \cap Z(a_{j+1};T)=\{0\}$.
- Let W be a T-admissible, proper T-invariant subspace of V. Let us try to find a s.t. $W \cap Z(a;T)=\{0\}$.

- Choose b not in W.
- T-conductor ideal is s(b;W)={g in F[x]|g(b) in W}
- Let f be the monic generator.
- f(T)b is in W.
- If W is T-admissible, there exists c in W s.t. f(T)b=f(T)c. ---(*).
- Let a = b-c. b-a is in W.
- Any g in F[x], g(T)b in W <-> g(T)a is in W:
  - g(T)(b-c)=g(T)b-g(T)c., g(T)b=g(T)a+g(T)c.

- Thus, S(a;W)=S(b;W).
- f(T)a = 0 by (*) for f the above T-conductor of b in W.
- g(T)a=0 <-> g(T)a in W for any g in F[x].
  - (->) clear.
  - (<-) g has to be in S(a;W). Thus g=hf for h in F[x]. g(T)a=h(T)f(T)a=0.
- Therefore, Z(a;T) $\cap$ W={0}. We found our vector a.

Cyclic decomposition theorem

- Theorem 3. T in L(V,V), V n-dim v.s. W_0 proper T-admissible subspace. Then
  - there exists nonzero $a_1,\ldots,a_r$ in V and
  - respective T-annihilators $p_1,\ldots,p_r$
  - such that (i) $V=W_0 \oplus Z(a_1;T) \oplus \ldots \oplus Z(a_r;T)$
  - (ii) p_k divides p_{k-1}, k=2,..,r.
  - Furthermore, r, $p_1,\ldots,p_r$ uniquely determined by (I),(ii) and $a_i \neq 0$. (a_i are not nec. unique).

- The proof will be not given here. But uses the Fact.
- One should try to follow it at least once.
- We will learn how to find a_is by examples.
- After a year or so, the proof might not seem so hard.
- Learning everything as if one prepares for exam is not the best way to learn.
- One needs to expand one's capabilities by forcing one self to do difficult tasks.

- Corollary. If T is a linear operator on Vn, then every T-admissible subspace has a complementary subspace which is invariant under T.
- Proof: W_0 T-inv. T-admissible. Assume W_0 is proper.
  - Let W_0' be $Z(a_1;T) \oplus \ldots \oplus Z(a_r;T)$ from Theorem 3.
  - Then W_0' is T-invariant and is complementary to W_0.

- Corollary. T linear operator V.
  - (a) There exists a in V s.t. T-annihilator of a =minpoly T.
  - (b) T has a cyclic vector <-> minpoly for T agrees with charpoly T.
- Proof:
  - (a) Let $W_0=\{0\}$. Then $V=Z(a_1;T) \oplus \ldots \oplus Z(a_r;T)$.
  - Since p_i all divides p_1, $p_1(T)(a_i)=0$ for all i and $p_1(T)=0$. p_1 is in Ann(T).
  - p_1 is the minimal degree monic poly killing a_1. Elements of Ann(T) also kills a_1.
  - p_1 is the minimal degree monic polynomial of Ann(T).
  - p_1 is the minimal polynomial of T.

- (b) (->) done before
- (<-) charpolyT=minpolyT= p_1 for a_1.
- degree minpoly T = n=dim V.
- n= dim $Z(a_1;T)$=degree p_1.
- $Z(a_1;T)=V$ and a_1 is a cyclic vector.

- Generalized Cayley-Hamilton theorem. T in L(V,V). Minimal poly p, charpoly f.
  - (i) p divides f.
  - (ii) p and f have the same factors.
  - (iii) If $p=f_1^{r\_1}\ldots.f_k^{r\_k}$, then $f= f_1^{d\_1}\ldots.f_k^{d\_k}$. d_i= nullity $f_i(T)^{r\_i}$/deg f_i.
- proof: omit.
- This tells you how to compute r_is
- And hence let you compute the minimal polynomial.

Rational forms

- Let $B_i=\{a_i, Ta_i, \ldots, T^{k\_i-1}a_i\}$ basis for $Z(a_i;T)$.
- k_i = dim $Z(a_i;T)$=deg p_i=deg Annihilator of a_i.
- Let $B=\{B_1, \ldots, B_r\}$.
- $[T]_B$=A=

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{bmatrix}$$

- A_i is a $k_i x k_i$-companion matrix of B_i.

$$A_i = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & \dots & 0 & -c_0 \\ 1 & 0 & 0 & 0 & \dots & \dots & 0 & -c_1 \\ 0 & 1 & 0 & 0 & \dots & \dots & 0 & -c_2 \\ 0 & 0 & 1 & 0 & \dots & \dots & 0 & -c_3 \\ 0 & 0 & 0 & 1 & \dots & \dots & 0 & -c_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & -c_{k-1} \end{bmatrix}$$

- Theorem 5. B nxn matrix over F. Then B is similar to one and only one matrix in a rational form.
- Proof: Omit.

- The char.polyT =char.polyA_1….char.polyA_r =p_1…p_r.:
  - char.polyA_i=p_i.
    - This follows since on Z(a_i;T), there is a cyclic vector a_i, and thus char.polyT_i=minpolyT_i=p_i.
- p_i is said to be an invariant factor.
- Note charpolyT/minpolyT=p_2…p_r.
- The computations of the invariant factors will be the subject of Section 7.4.

Examples

- Example 2: V 2-dim.v.s. over F. T:V->V linear operator. The possible cyclic subspace decompositions:
  - Case (i) minpoly p for T has degree 2.
    - Minpoly p=charpoly f and T has a cyclic vector.
    - If $p=x^2+c_1x+c_0$. Then the companion matrix is of the form:

$$\begin{bmatrix} 0 & -c_0 \\ 1 & -c_1 \end{bmatrix}$$

- (ii) minpoly p for T has degree 1. i.e., T=cI. for c a constant.
- Then there exists a1 and a2 in V s.t. $V=Z(a_1;T)\oplus Z(a_2;T)$. 1-dimensional spaces.
- p_1, p_2 T-annihilators of a_1 and a_2 of degree 1.
- Since p_2 divides the minimal poly $p_1=(x-c)$, $p_2=x-c$ also.
- This is a diagonalizable case.

$$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

- Example 3: $T:R^3 \to R^3$ linear operator given by $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ in the standard basis.
  - charpolyT=f=$(x-1)(x-2)^2$
  - minpolyT=p=(x-1)(x-2) (computed earlier)
  - Since $f=pp_2$, $p_2=(x-2)$.
  - There exists a_1 in V s.t. T-annihilator of a_1 is p and generate a cyclic space of dim 2 and there exists a_2 s.t. T-annihilator of a_2 is (x-2) and has a cyclic space of dim 1.

- The matrix A is similar to B= $\begin{bmatrix} 0 & -2 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

  (using companion matrices)

- Question? How to find a_1 and a_2?
  - In general, almost all vector will be a_1. (actually choose s.t deg $s(a_1;W)$ is maximal.)
  - Let $e_1=(1,0,0)$. Then $Te_1=(5,-1,3)$ is not in the span $\langle e_1 \rangle$.
  - Thus, $Z(e_1;T)$ has dim 2 $=\{a(1,0,0)+b(5,-1,3)|a,b \text{ in } R\}=\{(a+5b,-b,3b)|a,b, \text{ in } R\} =\{(x_1,x_2,x_3)|x_3=-3x_2\}$.
  - $Z(a_2:T)$ is null(T-2I) since $p_2=(x-2)$ and has dim 1.
  - Let $a_2=(2,1,0)$ an eigenvector.

- Now we use basis (e_1, Te_1, a_2). Then the change of basis matrix is S= $\begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & 0 \end{bmatrix}$
- Then $B=S^{-1}AS$.
- Example 4: T diagonalizable V->V with char.values c_1, c_2, c_3. $V=V_1 \oplus V_2 \oplus V_3$. Suppose $\dim V_1=1$, $\dim V_2=2$, $\dim V_3=3$. Then char $f=(x-c_1)(x-c_2)^2(x-c_3)^3$.

  Let us find a cyclic decomposition for T.

- Let a in V. Then a = $b_1+b_2+b_3$. $Tb_i=c_ib_i$.
- $f(T)a=f(c_1)b_1+f(c_2)b_2+f(c_3)b_3$.
- By Lagrange theorem for any (t_1,t_2,t_3), There is a polynomial f s.t. $f(c_i)=t_i, i=1,2,3$.
- Thus $Z(a;T) = \langle b_1,b_2,b_3 \rangle$.
- $f(T)a=0$ <-> $f(c_i)b_i=0$ for i=1,2,3.
- <-> $f(c_i)=0$ for all i s.t. $b_i \neq 0$.
- Thus, Ann(a)= $\prod_{b_i \neq 0} (x - c_i)$
- Let $B=\{b^1{}_1, b^2{}_1, b^2{}_2, b^3{}_1, b^3{}_2, b^3{}_3\}$.

- Define $a_1 = b^1{}_1+b^2{}_1+b^3{}_1$. $a_2=b^2{}_2+b^3{}_2$, $a_3=b^3{}_3$.
- $Z(a_1;T)=< b^1{}_1,b^2{}_1,b^3{}_1>$ $p_1=(x-c_1)(x-c_2)(x-c_3)$.
- $Z(a_2;T)=< b^2{}_2,b^3{}_2 >$, $p_2=(x-c_2)(x-c_3)$.
- $Z(a_3;T)= <b^3{}_3>$, $p_3=(x-c_3)$.
- $V= Z(a_1;T)\oplus Z(a_2;T)\oplus Z(a_3;T)$

- Another example T diagonalizable.
- $F=(x-1)^3(x-2)^4(x-3)^5$. $d_1=3, d_2=4, d_3=5$.
- Basis $\{b_1^1, b_2^1, b_3^1, b_1^2, b_2^2, b_3^2\ b_4^2, b_1^3, b_2^3\ , b_3^3, b_4^3, b_5^3\}$
- Define $$a_j := \sum_{d_i \geq j} b_j^i$$
- Then $Z(a_j;T)=\langle b_j^i \rangle$, $d_i \geq j$. and
- $T\text{-ann}(a_j) = p_j = \prod_{d_i \geq j} (x - c_i)$
- $V= Z(a_1;T) \oplus Z(a_2;T) \oplus \ldots \oplus Z(a_5;T)$

$$
\begin{aligned}
a_1 &= b_1^1 + b_1^2 + b_1^3 \\
a_2 &= b_2^1 + b_2^2 + b_2^3 \\
a_3 &= b_3^1 + b_3^2 + b_3^3 \\
a_4 &= \phantom{b_4^1 + {}} b_4^2 + b_4^3 \\
a_5 &= \phantom{b_5^1 + b_5^2 + {}} b_5^3
\end{aligned}
$$