# 7.3. Jordan form

Canonical form for matrices and transformations.

- N nilpotent on $V^n$.
- Cyclic decomposition $V=Z(a_1;N)\oplus\ldots\oplus Z(a_r;N)$.
  $p_1,\ldots,p_r$ N-annihilators,
  $p_{i+1}|p_i$ , $i=1,\ldots,r-1$.
- minpoly $N= x^k$, $k\leq n$:
  - $N^r=0$ for some r. Thus $x^r$ is in Ann(N).
  - minpoly N divides $x^r$ and hence minpolyN=$x^k$ for some k.
  - minpolyN divides charpoly N of deg $\leq n$.
- $p_i=x^{k\_i}$, -> $k=k_1 \geq k_2\geq\ldots\geq k_r\geq 1$.

- We now have a rational form for N:
  - Companion matrix of $x^{k\_i}$: $k_i$ x$k_i$-matrix

$$A_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

  - Thus the rational form of N is 1s at one below the diagonal that skips one space after $k_1$-1,$k_2$-1,…, so on.

- N is a direct sum of elementary nilpotent matrices
- A nilpotent nxn-matrix up to similarity <-> positive integers r, $k_1,\ldots,k_r$, such that $k_1+\ldots+k_r=n$, $k_i \geq k_{i+1}$
  - Proof is omitted.
- r=nullity N. In fact null $N=\langle N^{k\_i-1}a_i : i=1,\ldots,r\rangle$. $a_i$ cyclic vectors.

  Proof: a in V ->
  - $a=f_1(N)a_1+\ldots+f_r(N)a_r$, $f_i$ in F[x], deg $f_i \leq k_i$.

- $Na=0$ implies $N(f_i(N)a_i)=0$ for each i.
  - (N-invariant direct sum property)
- $xf_i(N)\ a_i = 0$.
- $xf_i$ is in N-annihilator of $a_i$
- $xf_i$ is divisible by $x^{k\_i}$.
- Thus, $f_i=c_ix^{k\_i-1}$, $c_i$ in F and $a=c_1(x^{k\_1-1})(N)a_1+\ldots.+c_r(x^{k\_r-1})(N)a_r$.
- Therefore $\{N^{k\_1-1}a_1, \ldots, N^{k\_r-1}a_r\}$ is a basis of null N, and dim = r.

- Jordan form construction:
- Let T be a linear operator where charpoly $f=(x-c_1)^{d\_1}\ldots.(x-c_k)^{d\_k}$. For example when F=C.
- minpoly $f= (x-c_1)^{r\_1}\ldots.(x-c_k)^{r\_k}$.
- Let $W_i$ = null $(T-c_iI)^{r\_i}$.
- $V=W_1\oplus\ldots\oplus W_r$.
- Let $T_i=T|W_i:W_i\text{->}W_i$.
- $N_i=T_i-c_iI$ is nilpotent.
- Choose a basis $B_i$ of $W_i$ s.t. $N_i$ is in rational form.
- Then $T_i= N_i+c_iI$.

- $[T_i]_{B\_i} =$ $$\begin{bmatrix} c_i & 0 & \cdots & 0 & 0 \\ 1 & c_i & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_i \end{bmatrix}$$

- There are some gaps of 1s here.
- If there are no gaps, then it is called the elementary Jordan matrix with char value $c_i$.

- A = $\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$ Jordan form

- $A_i$ = $\begin{bmatrix} J_1^i & 0 & \cdots & 0 \\ 0 & J_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_i}^i \end{bmatrix}$ elementary Jordan matrices

The size of elementary Jordan matrices decreases.

- Uniquenss of Jordan form:
- Suppose T is represented by a Jordan matrix. $V=W_1\oplus\ldots\oplus W_k$.
  - $A_i$ is $d_i$x$d_i$-matrix. $A_i$ on $W_i$.
  - Then charpolyT=$(x-c_1)^{d\_1}\ldots.(x-c_k)^{d\_k}$.
  - $c_1,\ldots,c_k$, $d_1,\ldots,d_k$ are determined unique up to order.
  - $W_i=\text{null}(T-c_iI)^{d\_i}$ clearly.
  - $A_i$ is uniquely determined by the rational form for $(T-c_iI)$. (Recall rational form is uniquely determined.)

- Properties of Jordan matrix:
- (1) $c_1,..,c_k$ distinct char.values. $c_i$ is repeated $d_i$=dim$W_i$=multiplicity in char poly of A.
- (2) $A_i$ direct sum of $n_i$ elementrary Jordan matrices. $n_i \leq d_i$.
- (3) $J^i_1$ of $A_i$ is $r_i$x$r_i$-matrix. $r_i$ is multiplicity of the minimal polynomial of T. (So we can read the minimal polynomial from the Jordan form)

- Proof of (3):
  - We show that $(T-c_1 I)^{r_1} \ldots (T-c_k I)^{r_k}=0$.
    - $T|W_i=A_i$ and $(A_i-c_i I)^{r_i}=0$ on $W_i$.
  - $(x-c_1)^{r_1} \ldots (x-c_k)^{r_k}$ is the least degree.
    - This can be seen from the largest elementary Jordan matrix $J^1_i$.
    - $(J^1_i - c_i I)^q \neq 0$ for $q < r_i$.
  - This completes the proof.