Linear Algebra: Midterm Exam (2007 Spring)

Justify your answers fully.

1. Let Q be the field of rational numbers.

   (a) (15pts.) Show that
   $$Q + Q\sqrt{2} = \{a + b\sqrt{2} | a, b \in Q\}$$
   is a vector space over Q. Find a basis and the dimension.

   (b) (15pts.) Show that $Q + Q\sqrt{2}$ is a field and a vector space over itself $Q + Q\sqrt{2}$.

   (c) (15pts.) Show that the set G of 2×2-complex Hermitian matrices is a vector space over $\mathbf{R}$. Find a basis and a dimension.

   (d) (15pts.) Is G a field when the multiplication is given by a matrix multiplication?

2. (25pts.) Prove: Let W be a subspace of a finite-dimensional vector space V over a field F and if $\{g_1, \ldots, g_r\}$ is a basis for W^0, then
$$W = \bigcap_{i=1}^{r} N_{g_i}.$$

3. Prove or disprove whether each of the following sets is an ideal in $\mathbf{C}[x]$.

   (a) (10 pts.)
   $$M = \{f | f \in \mathbf{C}[x] \text{ is a polynomial of odd degrees.}\}$$

   (b) (10 pts.)
   $$M = \{f | f \in \mathbf{C}[x] \text{ is a polynomial of degrees} \geq 3 \text{ or is zero.}\}$$

   (c) (10 pts.)
   $$M = \{f \in \mathbf{C}[x] | \, f(2) + f(4) = 0\}$$

   (d) (10 pts.)
   $$M = \{f \in \mathbf{C}[x] | \, f(2)f(4) = 0\}$$

4. Let F be a field of characteristic 0.

   (a) (10pts.) State the Taylor's formular for $f \in F[x]$.

   (b) (15pts.) Prove that if $f \in F[x]$ and $D^k f(c) = 0$ for $k = 0, 1, 2, \ldots, r-1$ and $D^r f(c) \neq 0$. Then c is a root of multiplicity r.