Chapter 7

Section 1: Rational Forms

7.1. Rational forms

- Definition: T in L(V,V), a vector *a*.
  T-cyclic subspace generated by *a* is
  $Z(a;T)=\{v=g(T)a | g \text{ in } F[x]\}$.
- $Z(a;T)=<a, Ta, T^2a, \ldots.>$
- If $Z(a:T)=V$, then *a* is said to be a cyclic vector for T.
- Recall T-annihilator of *a* is the ideal
  $M(a:T)=<g \text{ in } F[x] | g(T)a=0>=p_aF[x]$.
- p_a is the T-annihilator of *a.*

- Theorem 1. $a \neq 0$. p_a T-annihilator of a.
  - (i) deg p_a = dim Z(a;T).
  - (ii) If deg p_a =k, a, Ta,…,$T^{k-1}a$ is a basis of
  - (iii) Let U:=T|Z(a;T):Z(a:T)->Z(a;T). Minpoly U=p_a.
- Proof: Let g in F[x]. g=p_aq+r. deg(r) < deg(p_a). g(T)a=r(T)a.
  - r(T)a is a linear combinations of a, Ta,…,$T^{k-1}a$.
  - Thus, this k vectors span Z(a;T).
  - They are linearly independent. Otherwise, we get another g of lower than k degree s.t. g(T)a =0.
  - (i),(ii) are proved.

- $U := T|Z(a;T) : Z(a:T) \to Z(a;T)$.
- g in F[x].
- $p_a(U)g(T)a = p_a(T)g(T)a$ (since g(T) a is in $Z(a;T)$.)

  $= g(T)p_a(T)a = g(T)0 = 0$.
- $p_a(U)=0$ on $Z(a;T)$ and p_a is monic.
- If h is a polynomial of lower-degree than p_a, then $h(U)\neq 0$. (since $h(U)a=h(T)a\neq 0$).
- Thus, p_a is the minimal polynomial of U.

- Suppose T has a cyclic vector *a*.
- deg minpolyU=dimZ(*a*;T)=dim V=n.
- minpoly U=minpoly T. (minpoly T is in S(a;{0}) and divisible by p_a.)
- Thus, minpoly T = char.poly T.
- We obtain:

T has a cyclic vector <-> minpoly T=char.polyT.

- Proof: (->) done above.
  - (<-) Later, we show for any T, there is a vector v s.t. minpolyT=annihilator v. (p.237. Corollary).
  - So if minpolyT=charpolyT. Then dimZ(v;T)=n and v is a cyclic vector.

- Study T by cyclic vector.
- U on W with a cyclic vector v. (W=Z(v:T) for example and U the restriction of T.)
- $v, Uv, U^2v,\ldots,U^{k-1}v$ is a basis of W.
- U-annihiltor of v = minpoly U by Theorem 1.
- Let $v_i=U^{i-1}v$. $i=1,\ldots,k$.
- Let $B=\{v_1,\ldots,v_k\}$.
- $Uv_i=v_{i+1}$. $i=1,\ldots,k-1$.
- $Uv_k=-c_0v_1-c_1v_2-\ldots-c_{k-1}v_k$ where $\text{minpoly}U=c_0+c_1x+\ldots+c_{k-1}x^{k-1}+x^k$.
  - $(c_0v+c_1Uv+\ldots+c_{k-1}U^{k-1}v+U^kv=0.)$

$$[U]_B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 & -c_2 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & -c_3 \\ 0 & 0 & 0 & 1 & \cdots & \cdots & 0 & -c_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -c_{k-1} \end{bmatrix}$$

- This is called the companion matrix of p_a. (defined for any *monic* polynomial.)

- Theorem 2. If U is a linear operator on a f.d.v.s.W, then U has a cyclic vector iff there is some ordered basis where U is represented by a companion matrix.
- Proof: (->) Done above.
- (<-) If we have a basis $\{v_1,\ldots,v_k\}$,
  - then v_1 is the cyclic vector.

- Corollary. If A is the companion matrix of a monic polynomial p, then p is both the minimal and the characteristic polynomial of A.
- Proof: Let *a*=(1,0,…0). Then a is a cyclic vector and Z(a;A)=V.
  - The annihilator of *a* is p. deg p=n also.
  - By Theorem 1(iii), the minimal poly for T is p.
  - Since p divides char.polyA. And p has degree n. p=char.polyA.