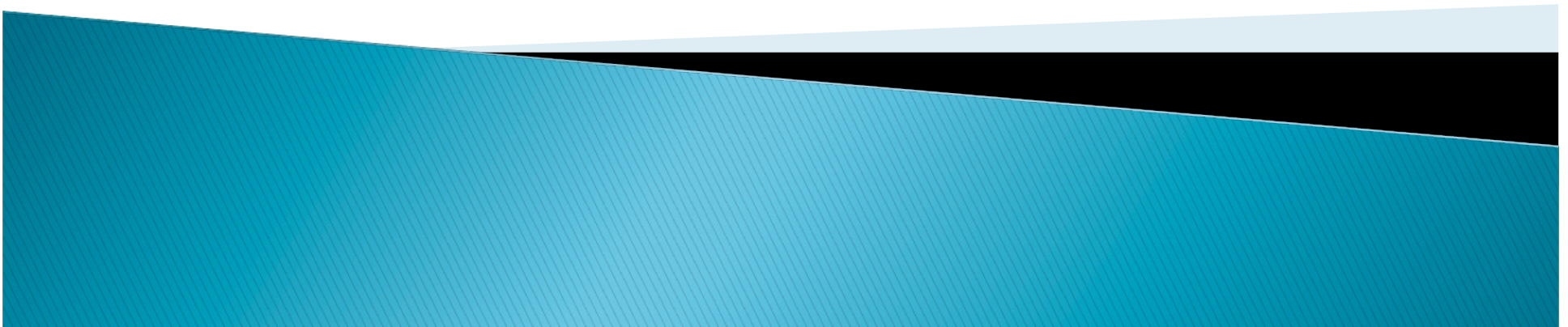


4.2. Properties of determinant



Determinant of A^T

- ▶ For 2x2 matrix $\det(A)=\det(A^T)$.
- ▶ In general we have

Theorem 4.2.1 *If A is a square matrix, then $\det(A) = \det(A^T)$.*

- ▶ The simplest way to prove this is to use the formula.
- ▶ The another method is to use the cofactor expansion along rows for A and that along columns for A^T . See p 190-191.



Effect of elementary operations on a determinant.

- ▶ The following will be important in computing:

Theorem 4.2.2 *Let A be an $n \times n$ matrix.*

- (a) *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k , then $\det(B) = k \det(A)$.*
- (b) *If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.*
- (c) *If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.*



▶ Proof (a):

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}.$$

- If we multiply the i th-row by k , then each term in $\det(A)$ get multiplied by k .

▶ Proof (b): We can use formula.

- Suppose we exchanged two columns. Then in each elementary products in $\det(B)$.
- We can make a one-to-one correspondence between elementary products in $\det(A)$ to those of $\det(B)$ by identifying the same term up to signs.
- The sign in each term of B should be reversed from the corresponding one in A .
- To see in case we exchange two rows, we use A^T .



- ▶ Proof (c): Add i -th row to j -th row. Cofactor expand $\det(A)$ along the j -th row. Then we have

$$\begin{aligned}\det(A') &= (a_{j1} + ka_{i1})C_{j1} + (a_{j2} + ka_{i2})C_{j2} + \dots + (a_{jn} + ka_{in})C_{jn} \\ &= \det(A) + k \det(A'')\end{aligned}$$

- Here A'' is a matrix obtained by replacing the j -th row of A by the i -th row of A .
- By Theorem 4.2.3 (a), $\det(A'')=0$.
- For column case, we use A^T .
- ▶ See Example 1.



Theorem 4.2.3 *Let A be an $n \times n$ matrix.*

- (a) *If A has two identical rows or columns, then $\det(A) = 0$.*
- (b) *If A has two proportional rows or columns, then $\det(A) = 0$.*
- (c) $\det(kA) = k^n \det(A)$.

- ▶ **Proof (a):** If A has two same rows, then after the exchange of the two rows, we still get A . By Theorem 4.2.2 (b), $\det(A) = -\det(A)$. Thus $\det(A) = 0$.
- ▶ **Proof (b):** If A has two proportional rows, then one row is a multiple of the other row, say by k . If we multiply the other row by $1/k$, then the result has determinant 0. Thus $\det(A) = 0$ by Theorem 4.2.2 (a).
- ▶ **Proof (c):** omit.

Simplifying cofactor expansion

- ▶ Given a matrix, we do row and column operations of type Theorem 4.2.2 (c) to make many zeros.
- ▶ Example 4.



Determinants by Gaussian eliminations

- ▶ We can use Gaussian elimination to evaluate a determinant.
- ▶ Each multiplication by k of a row should be compensated by multiplying by $1/k$ to the result.
- ▶ Each row exchange should be compensated by the multiplication by -1 .
- ▶ For type (c), we do not need any compensations.
- ▶ See Example *.



Theorem 4.2.4 *A square matrix A is invertible if and only if $\det(A) \neq 0$.*

- ▶ First we need. R ref of A . Then $\det(R)=0$ iff $\det(A)=0$. This follows since each elementary operation preserves \det being 0 or nonzero.
- ▶ Proof: \rightarrow) If A is invertible, then ref of A is I . Thus, $\det(A)$ is nonzero.
- ▶ \leftarrow) If $\det(A)$ is not zero, then $\det(R)$ is not zero for the ref R of A . Thus R has no zero rows. Hence R is I . If ref of A is I , then A is invertible by Theorem 3.3.3.



Theorem 4.2.5 *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A) \det(B) \quad (1)$$

▶ **Proof:** We need:

Lemma 4.2.8 *Let E be an $n \times n$ elementary matrix and I_n the $n \times n$ identity matrix.*

- (a) *If E results by multiplying a row of I_n by k , then $\det(E) = k$.*
- (b) *If E results by interchanging two rows of I_n , then $\det(E) = -1$.*
- (c) *If E results by adding a multiple of one row of I_n to another, then $\det(E) = 1$.*

Lemma 4.2.9 *If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EB) = \det(E) \det(B)$$



- ▶ Proof of 4.2.8: Just computations
- ▶ Proof of 4.2.9. EB is just a result of row operation. $\det(EB)$ is just some number times $\det(B)$. The number is $\det(E)$.

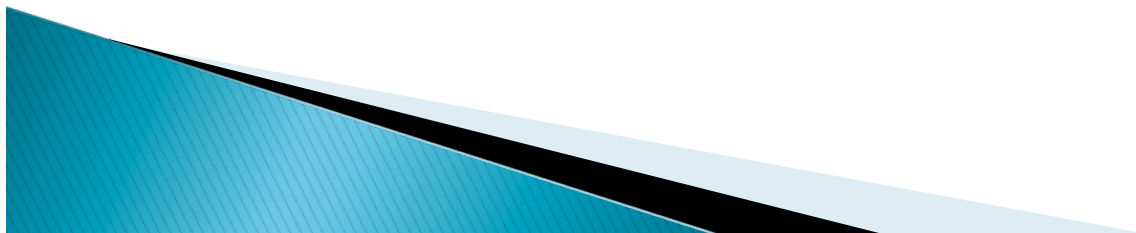
- ▶ Proof of 4.2.5: If A is singular (i.e. not invertible), then AB is singular (not invertible) also. By Theorem 4.2.4 both have determinant 0 and we are done.

- ▶ If A is invertible, then $A = E_1 E_2 \dots E_k$.
 - $\det(AB) = \det(E_1 E_2 \dots E_k B) = \det(E_1) \det(E_2 \dots E_k B) = \det(E_1) \det(E_2) \dots \det(E_k) \det(B)$.
 - $\det(A) = \det(E_1) \det(E_2) \dots \det(E_k)$.
 - Thus the conclusion holds.



Computing determinants by LU-decompositions

- ▶ $A=LU$. $\det(A)=\det(L)\det(U)$.
- ▶ We just need to multiply the diagonals.
- ▶ Obtaining LU decompositions is around $2n^3/3$ which is much smaller than $n!$.



Determinant of an inverse matrix

- ▶ Theorem 4.2.6. $\det(A^{-1}) = 1/\det(A)$.
- ▶ Proof: $AA^{-1} = I$. $\det(A)\det(A^{-1}) = \det(I) = 1$.
- ▶ Determinant of $A+B$.
 - It is not true that $\det(A+B) = \det(A) + \det(B)$.
 - However, there are other invariants that we haven't learned that we can compensate the difference.



A unifying theorem

Theorem 4.2.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

- (a) *The reduced row echelon form of A is I_n .*
- (b) *A is expressible as a product of elementary matrices.*
- (c) *A is invertible.*
- (d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- (e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector \mathbf{b} in R^n .*
- (f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector \mathbf{b} in R^n .*
- (g) *The column vectors of A are linearly independent.*
- (h) *The row vectors of A are linearly independent.*
- (i) *$\det(A) \neq 0$.*



Ex set 4.2.

- ▶ 1-10 Theory practise
- ▶ 11-18 Gaussian elimination
- ▶ 19-28 Theory

