4.2. Properties of determinant

Determinant of A^T

- For 2x2 matrix $\det(A)=\det(A^T)$.
- In general we have

Theorem 4.2.1 *If A is a square matrix, then* $\det(A) = \det(A^T)$.

- The simplest way to prove this is to use the formula.
- The another method is to use the cofactor expansion along rows for A and that along columns for A^T. See p 190-191.

Effect of elementary operations on a determinant.

- The following will be important in computing:

Theorem 4.2.2 *Let A be an $n \times n$ matrix.*

(*a*) *If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then* $\det(B) = k\det(A)$.

(*b*) *If B is the matrix that results when two rows or two columns of A are interchanged, then* $\det(B) = -\det(A)$.

(*c*) *If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then* $\det(B) = \det(A)$.

- Proof (a): $\det(A)=a_{i1}C_{i1}+a_{i2}C_{i2}+\ldots+a_{in}C_{in}$.
  - If we multiply the ith-row by k, then each term in det(A) get multiplied by k.
- Proof (b): We can use formula.
  - Suppose we exchanged two columns. Then in each elementary products in det(B).
  - We can make a one-to-one correspondence between elementary products in det(A) to those of det(B) by identifying the same term up to signs.
  - The sign in each term of B should be reversed from the corresponding one in A.
  - To see in case we exchange two rows, we use A^T.

- Proof (c): Add i-th row to j-th row. Cofactor expand det(A) along the j-th row. Then we have

$$\det(A') = (a_{j1} + ka_{i1})C_{j1} + (a_{j2} + ka_{i2})C_{j2} + \ldots + (a_{jn} + ka_{in})C_{jn}$$

$$= \det(A) + k\det(A'')$$

  - Here A'' is a matrix obtained by replacing the j-th row of A by the i-th row of A.
  - By Theorem 4.2.3 (a), det(A'')=0.
  - For column case, we use A^T.
- See Example 1.

Theorem 4.2.3 *Let A be an $n \times n$ matrix.*

(*a*) *If A has two identical rows or columns, then* $\det(A) = 0$.

(*b*) *If A has two proportional rows or columns, then* $\det(A) = 0$.

(*c*) $\det(kA) = k^n \det(A)$.

- Proof (a): If A has two same rows, then after the exchange of the two rows, we still get A.
  By Theorem 4.2.2 (b), det(A)=-det(A). Thus det(A)=0.
- Proof (b): If A has two proportional rows, then one row is a multiple of the other row, say by k. If we multiply the other row by 1/k, then the result has determinant 0. Thus det(A)=0 by Theorem 4.2.2 (a).
- Proof (c): omit.

Simplifying cofactor expansion

- Given a matrix, we do row and column operations of type Theorem 4.2.2 (c) to make many zeros.
- Example 4.

Determinats by Gaussian eliminations

- We can use Gaussian elimination to evaluate a determinant.
- Each multiplication by k of a row should be compensated by multiplying by1/k to the result.
- Each row exchange should be compensated by the multiplication by -1.
- For type (c), we do not need any compensations.
- See Example *.

Theorem 4.2.4 *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

- First we need. R ref of A. Then det(R)=0 iff det(A)=0. This follows since each elementary operation preserves det being 0 or nonzero.
- Proof: ->) If A is invertible, then ref of A is I. Thus, det(A) is nonzero.
- <-) If det(A) is not zero, then det(R) is not zero for the ref R of A. Thus R has no zero rows. Hence R is I. If ref of A is I, then A is invertible by Theorem 3.3.3.

Theorem 4.2.5 *If A and B are square matrices of the same size, then*

$$\det(AB) = \det(A)\det(B) \tag{1}$$

- Proof: We need:

Lemma 4.2.8 *Let E be an $n \times n$ elementary matrix and I_n the $n \times n$ identity matrix.*

(*a*) *If E results by multiplying a row of I_n by k, then* $\det(E) = k$.

(*b*) *If E results by interchanging two rows of I_n, then* $\det(E) = -1$.

(*c*) *If E results by adding a multiple of one row of I_n to another, then* $\det(E) = 1$.

Lemma 4.2.9 *If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then*

$$\det(EB) = \det(E)\det(B)$$

- Proof of 4.2.8: Just computations
- Proof of 4.2.9. EB is just a result of row operation. det(EB) is just some number times det(B).
  The number is det(E).

- Proof of 4.2.5: If A is singular (i.e. not invertible), then AB is singular (not invertible) also. By Theorem 4.2.4 both have determinant 0 and we are done.

- If A is invertible, then $A=E_1E_2\ldots E_k$.
  - $\det(AB)=\det(E_1E_2\ldots E_kB)= \det(E_1)\det(E_2\ldots E_kB) = \det(E_1)\det(E_2)\ldots\det(E_k)\det(B)$.
  - $\det(A)=\det(E_1)\det(E_2)\ldots\det(E_k)$.
  - Thus the conclusion holds.

Computing determinants by LU-decompositions

- A=LU. det(A)=det(L)det(U).
- We just need to multiply the diagonals.
- Obtaining LU decompostions is around $2n^3/3$ which is much smaller than n!.

Determinant of an inverse matrix

- Theorem 4.2.6. $\det(A^{-1}) = 1/\det(A)$.
- Proof: $AA^{-1}=I$. $\det(A)\det(A^{-1})=\det(I)=1$.
- Deteminant of A+B.
  - It is not true that $\det(A+B)=\det(A)+\det(B)$.
  - However, there are other invariants that we haven't learned that we can compensate the difference.

A unifying theorem

Theorem 4.2.7 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in R^n.*

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in R^n.*

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

(*i*) $\det(A) \neq 0$.

Ex set 4.2.

- 1-10 Theory practise
- 11-18 Gaussian elimination
- 19-28 Theory