4.3. Cramer’s Rules and applications of determinant

MATRIX OF COFACTORS OF A

- C_ij cofactor of A_ij

Definition 4.3.2 If A is an $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors*** from A. The transpose of this matrix is called the ***adjoint*** (or sometimes the ***adjugate***) of A and is denoted by $\text{adj}(A)$.

Theorem 4.3.1 *If the entries in any row (column) of a square matrix are multiplied by the cofactors of the corresponding entries in a different row (column), then the sum of the products is zero.*

- Proof: We take an i-th row and copy it over the j—th row.
- The resulting matrix A' has two equal rows. Hence the determinant is zero.
- The cofactor expansion of A' over the j-th row is the above expression.
- Example 1:

FORMULA INVERSE MATRIX

Theorem 4.3.3 *If A is an invertible matrix, then*

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) \qquad (2)$$

- Proof: We show A.adj(A) = det(A)I.
  - The reason is that row i times column j gives 0 if i does not equal j by 4.3.1.
  - If i=j, then row times the column is det(A).
  - Hence the result is det(A) on the diagonal and zero elsewhere.

- If an integer matrix has a determinant ±1, then its inverse is another integer matrix.
- To see this, adj(A) is an integer matrix.
- Now multiply an integer by 1/det(A).
- This is sometimes useful to know in group theory.

CRAMER'S RULE

Theorem 4.3.4 (*Cramer's Rule*) *If* $A\mathbf{x} = \mathbf{b}$ *is a linear system of n equations in n unknowns, then the system has a unique solution if and only if* $\det(A) \neq 0$, *in which case the solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix that results when the* jth *column of* A *is replaced by* $\mathbf{b}$.

- Proof : Omit.
- See Example 6.

GEOMETRIC INTERPRETATION OF DET(A)

- In the plane, the area of a parallelogram spanned by vectors u,v is given by |det[u,v]|.
- Proof: det[u,v]=det[u,v-P_u(v)] where P_u(v) is a projection v to the line containing u.
  - Now the columns are perpendicular.
  - For perpendicular x,y, |det[x,y]|=||x||||y|| since y is obtained by taking coordinates of x and changing the order and a scalar multiplication.
  - |det[u,v-P_u(v)]| equals the product of lengths. That is the area of the parallelogram.

Theorem 4.3.5

(*a*) *If A is a* 2×2 *matrix, then* $|\det(A)|$ *represents the area of the parallelogram determined by the two column vectors of A when they are positioned so their initial points coincide.*

(*b*) *If A is a* 3×3 *matrix, then* $|\det(A)|$ *represents the volume of the parallelepiped determined by the three column vectors of A when they are positioned so their initial points coincide.*

Theorem 4.3.6 *Suppose that a triangle in the xy-plane has vertices* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, *and* $P_3(x_3, y_3)$ *and that the labeling is such that the triangle is traversed counterclockwise from* P_1 *to* P_2 *to* P_3. *Then the area of the triangle is given by*

$$\text{area } \triangle P_1 P_2 P_3 = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{7}$$

VANDERMOND DETERMINANT

- A_ij = x_i^{j-1}. A is called a Vandermond matrix.
  - The determinant is called the Vandermond determinant.
  - The values is the product of all (x_j-x_i) where j>i for i,j in 1,2,…,n.

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix} = \Pi_{1\le i<j\le n}(x_j - x_i)$$

  - Thus if {x_1,x_2,…,x_n} are a set of mutually distinct points, then the determinant is not zero.

CROSS PRODUCT

- Cross products useful in mechanics of spinning objects, electromagnetism…
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Definition 4.3.7 If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in R^3, then the ***cross product of* u *with* v**, denoted by $\mathbf{u} \times \mathbf{v}$, is the vector in R^3 defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \tag{10}$$

or equivalently,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \tag{11}$$

CALCULATIONS

- We let i=(1,0,0),j=(0,1,0),k=(0,0,1).

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_{31} \end{vmatrix}$$

- Example 9.

Properties:

Theorem 4.3.8 *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* R^3 *and* k *is a scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Theorem 4.3.9 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* R^3, *then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u x v is orthogonal to u]**

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ **[u x v is orthogonal to v]**

- Actually, w.(uxv) is called the vector triple product and it equals the determinant of a 3x3 matrix with rows w,u,v.
- Thus u.(uxv)=0, v.(uxv)=0 since the matrix has two rows equal.
  - This means that uxv is orthogonal to u and v.
  - We need to use the right hand rule. See Fig 4.3.5.
- i, j, k satisfy interesting relations:
  - ixj=k, jxk=i, kxi=j
  - jxi=-k, kxj=-i, ixk=-j.

- The cross product is not commutative (actually anticommutative) and not associative.
- ix(jxj)=ix0=0. (ixj)xj= kxj=i.

Theorem 4.3.10 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in* R^3*, and let* θ *be the angle between these vectors.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$

(*b*) *The area A of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\| \tag{14}$$