Chapter 6. Linear transformations

The purpose is to understand linear transformations , see various examples, kernel range, compositions and invertibility

6.1. Matrices as transformations

Definition 6.1.1 Given a set D of allowable inputs, a ***function*** f is a rule that associates a unique output with each input from D; the set D is called the ***domain*** of f. If the input is denoted by x, then the corresponding output is denoted by $f(x)$ (read, "f of x"). The output is also called the ***value*** of f at x or the ***image*** of x under f, and we say that f ***maps*** x into $f(x)$. It is common to denote the output by the single letter y and write $y = f(x)$. The set of all outputs y that results as x varies over the domain is called the ***range*** of f.

- A function is a set {(x, f(x))|x in D} where x=y means f(x)=f(y)
- Example:
  - T(x_1,x_2)=(x_1,x_2) or the identity map.
  - T(x_1, x_2)=(c_1,c_2) or a constant map.

- Example: T: R^3->R^3. T(x_1,x_2,x_3)=(x_1x_2,x_2x_3,x_3x_1).
- Example: Given 2x3 matrix A=[[1,0,1],[0,2,1]], define T(x_1,x_2,x_3) = (x_1+x_3, 2x_2+x_3). Or T_A(x)=Ax.
- Given a transformation T: R^n->R^m. A domain is R^n and codomain is R^m. The range is the actual set T(R^n) in R^m which may or may not be the whole of R^m.
- An operator is a transformation R^n->R^n.

Matrix transformation

- Given A mxn matrix.
- We define T_A:R^n -> R^m by x -> Ax or T(x)=Ax.
- T_A: multiplication by A, or transformation A.
- A matrix transformation and the matrix itself is often considered a same object.
- Example: zero transformation T_O(x)=Ox=O.
- Identity operator T_I(x)=Ix=x.

Linear transformation

- The term linear was used to denote that the order of a polynomial was no more than one.
- Here, we will change meaning somewhat.
- A transformation will be linear if it sends O to O and each line to a line and planes to planes and so on.
- It turns out that this means that the transformation preserves addition and scalar multiplications and conversely.

- Superposition principle: T(c_1v_1+c_2v_2+…+c_kv_k) = c_1T(v_1)+c_2T(v_2)+…+c_kT(v_k).
- Actually this is linearity. Physicists use it in different way also.

Definition 6.1.2 A function $T : R^n \to R^m$ is called a ***linear transformation*** from R^n to R^m if the following two properties hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for all scalars c:

(i)	$T(c\mathbf{u}) = cT(\mathbf{u})$	**[Homogeneity property]**
(ii)	$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$	**[Additivity property]**

In the special case where $m = n$, the linear transformation T is called a ***linear operator*** on R^n.

- Example: matrix transformations are linear. $T_A(c_1x_1+c_2x_2)=A(c_1x_1+c_2x_2)=$ $c_1Ax_1+c_2Ax_2=c_1T(x_1)+c_2T(x_2)$.
- Example: 2nd or higher order transformations are nonlinear. They do not preserve the scalar multiplication or additions sometimes.
  - $T(x_1,x_2,x_3)=(x_1x_2,x_2x_3,x_3x_1)$.
  - $T(2x_1,2x_2,2x_3)=4T(x_1,x_2,x_3)$.
  - $T(x_1+x'_1,x_2+x'_2,x_3+x'_3) = ((x_1+x'_1)(x_2+x'_2),$ $(x_2+x'_2)(x_3+x'_3),(x_3+x'_3)(x_1+x'_1))$ is not $T(x_1,x_2,x_3)+T(x'_1,x'_2,x'_3)$ for arbitrary choices.

Properties

Theorem 6.1.3 *If* $T : R^n \rightarrow R^m$ *is a linear transformation, then:*

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{u}) = -T(\mathbf{u})$

(c) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

- Proof: (a) T(O)=T(0v)=0T(v)=O.
- Example: A translation is not linear.
  - T(x)=x+x_0. O-> x_0.

All linear transformations are matrix transformations

- Suppose that T is liner: R^n -> R^m.
  - x=x_1e_1+x_2e_2+…+x_ne_n.
  - T(x)=x_1T(e_1)+x_2T(e_2)+…+x_nT(e_n).
  - T(x)=[T(e_1),T(e_2),…,T(e_n)][x_1,x_2,…,x_n]^T.
  - Let A be [T(e_1),T(e_2),…,T(e_n)]. Then T(x)=Ax.

Theorem 6.1.4 *Let $T : R^n \to R^m$ be a linear transformation, and suppose that vectors are expressed in column form. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors in R^n, and if $\mathbf{x}$ is any vector in R^n, then $T(\mathbf{x})$ can be expressed as*

$$T(\mathbf{x}) = A\mathbf{x} \tag{13}$$

where

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

- A is a (standard) matrix corresponding to T.
- T is a transformation corresponding to A.
- T is a transformation represented by A.
- T is the transformation A.
- $A=[T]=[T(e_1),T(e_2),\ldots,T(e_n)]$.
- $T(x)=[T]x$.
- Example: $T(x)=cx$. c is some number. T is linear and is called a scaling operator.
- Then $[T]=cI$.

Representing transformations by equations….

- R^n coordinates $(x_1,x_2,\dots,x_n)$.
- R^m coordinates $(w_1,w_2,\dots, w_m)$
- Then $(w_1,w_2,\dots,w_m)=T(x_1,x_2,\dots,x_n)$ can be written:
  - $w_1= a_{11} x_1+a_{12} x_2+\dots+a_{1n} x_n$
  - $w_2= a_{21} x_1+a_{22} x_2+\dots+a_{2n} x_n$
  - …….
  - $w_m=a_{m1} x_1+a_{m2} x_2+\dots+a_{mn} x_n$.
- Conversely, this equation defines $w=Ax$ and hence a linear transformation T_A.
- We can consider these identical definitions.

Rotations about the origin.

- Let us make a transformation that preserves length and send a vector to a vector rotated by an angle θ.
- e_1 -> (cosθ, sinθ), e_2 -> (-sinθ,cosθ).
- Thus let [T] =[Te_1,Te_2]
  = [[cosθ,-sinθ],[sinθ,cosθ]].
- Thus R_θx= [[cosθ,-sinθ],[sinθ,cosθ]]x.
- A rotation about nonorigin is not linear.

Reflection about a line through the origin.

- Take a line L through the origin having angle $\theta$with the positive x-axis.
- $T(e_1)$ is length 1 and has angle $2\theta$with the positive x-axis. $T(e_1)=(\cos 2\theta,\sin 2\theta)$.
- $T(e_2)$ is length 1 and has angle $2(\pi/2-\theta)$ with the positive y-axis and has angle
  $(\pi/2-2\theta)$ with the positive x-axis.
  $T(e_2)=(\cos(\pi/2-2\theta),\sin(\pi/2-2\theta))=(\sin 2\theta,-\cos 2\theta)$.
- $H_\theta(x)=[[\cos 2\theta, \sin 2\theta],[\sin 2\theta,-\cos 2\theta]]x$

- Examples:
  - (a) $T(x,y)=(-y,x)$: reflection about the y-axis
  - (b) $T(x,y)=(x,-y)$: reflection about the x-axis.
  - (c) $T(x,y)=(y,x)$: reflection about y=x line.
- Example 13: $\theta=\pi/3$.
  - H_π/3(x)
    = [[cos(2π/3), sin(2π/3)],[sin(2π/3),-cos(2π/3)]]
    = [[-1/2,1/√3],[1/√3,1/2]]x.

Orthogonal projection onto the line through the origin.

- Define P_θ: R^2->R^2 by sending a point x to a line L through O with angle θ with the positive x-axis.
- We find the formula by P_θ(x)-x=(H_θ(x)-x)/2.
- Thus, P_θ(x)=H_θ(x)/2+x/2 = ½(H_θ+I)(x).
- P_θ=½(H_θ+I).

$$\begin{bmatrix} (1+\cos 2\theta)/2 & (\sin 2\theta)/2 \\ (\sin 2\theta)/2 & (1+\cos 2\theta)/2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

- The projection to the x-axis. $\Theta=0$. Thus the matrix is [[1,0],[0,0]]. (x,y)->(x,0).
- The projection to the y-axis. $\Theta=\pi/2$. Thus the matrix is [[0,0],[0,1]]. (x,y)-> (0, y).