6.3. KERNEL AND RANGE

Kernel of a linear transformation

- Kernel tells you how much is eliminated.

Definition 6.3.1 If $T : R^n \to R^m$ is a linear transformation, then the set of vectors in R^n that T maps into $\mathbf{0}$ is called the ***kernel*** of T and is denoted by $\ker(T)$.

- Example:
  - O-operator: Then R^n is the kernel.
  - Identity operator: {O} is the kernel.
  - Orthogonal projection to a plane: the perpendicular line through the origin.

Theorem 6.3.2 *If $T : R^n \to R^m$ is a linear transformation, then the kernel of T is a subspace of R^n.*

- Proof: We can do scalar multiplications and vector additions in the kernel.
- The kernel of a matrix transformation T_A is the set of x such that Ax=O.

Theorem 6.3.3 *If A is an $m \times n$ matrix, then the kernel of the corresponding linear transformation is the solution space of $A\mathbf{x} = \mathbf{0}$.*

Definition 6.3.4 If A is an $m \times n$ matrix, then the solution space of the linear system $A\mathbf{x} = \mathbf{0}$, or, equivalently, the kernel of the transformation T_A, is called the ***null space*** of the matrix A and is denoted by $\text{null}(A)$.

Theorem 6.3.5 *If $T : R^n \rightarrow R^m$ is a linear transformation, then T maps subspaces of R^n into subspaces of R^m.*

- Proof: This follows from the fact that T preserves additions and scalar multiplications.

Range of a linear transformation

Definition 6.3.6 If $T: R^n \to R^m$ is a linear transformation, then the ***range*** of T, denoted by $\text{ran}(T)$, is the set of all vectors in R^m that are images of at least one vector in R^n. Stated another way, $\text{ran}(T)$ is the image of the domain R^n under the transformation T.

- Examples:
  - For 0-operator: Range is {O}.
  - For Id: the range is R^m.
  - For orthogonal projections to a plane P: the rangle is the plane P.

Theorem 6.3.7 *If $T: R^n \to R^m$ is a linear transformation, then $\text{ran}(T)$ is a subspace of R^m.*

Range of a matrix transformation

- A mxn matrix
- T_A:R^n->R^m.
- T_A(x)=Ax.

Theorem 6.3.8 *If A is an $m \times n$ matrix, then the range of the corresponding linear transformation is the column space of A.*

- See Example 5.
- Example 6: To check whether some vector is in the range.

Existence and Uniqueness

- Existence question: Is every vector in the codomain of T in the range? (If not, which subspace is the range.)
- Uniqueness question: Can two vectors map to a same vector under T?

Definition 6.3.9 A transformation $T: R^n \to R^m$ is said to be ***onto*** if its range is the entire codomain R^m; that is, every vector in R^m is the image of at least one vector in R^n.

Definition 6.3.10 A transformation $T: R^n \to R^m$ is said to be ***one-to-one*** (sometimes written 1–1) if T maps distinct vectors in R^n into distinct vectors in R^m.

- Example: A rotation in R^2.
  - This is one-to-one since it has a nonsingular matrix.
  - This is also onto since the matrix has an inverse.
- Example: An orthogonal projection to a plane.
  - This is not one-to-one since many vectors go to O.
  - This is not onto since P is not all of the codomain.
- See Examples 9 and10.

Theorem 6.3.11 *If $T : R^n \rightarrow R^m$ is a linear transformation, then the following statements are equivalent.*

(a) *T is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

- Proof: (a)->(b). T(O)=O. If T(x)=O, then x=O since T is one-to-one. Thus Ker(T)={O}.
- (b)->(a). Suppose x_1 is not x_2. If T(x_1)=T(x_2), then T(x_1-x_2)=O. Thus, x_1-x_2=O as ker(T)={O}. Therefore x_1=x_2.

One to one and onto from linear systems.

- T_A(x)=0 <-> Ax=0.
- T_A(x)=b <-> Ax = b.

Theorem 6.3.12 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : R^n \to R^m$ is one-to-one if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Theorem 6.3.13 *If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : R^n \to R^m$ is onto if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in R^n.*

- These are solvable questions.

Theorem 6.3.14 *If $T: R^n \to R^n$ is a linear operator on R^n, then T is one-to-one if and only if it is onto.*

- Proof: Theorem 4.4.7 (d) and (e) are equivalent. (d) <-> one-to-one
- (e) <-> onto.

Theorem 6.3.15 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(*e*) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.*

(*f*) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.*

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

(*i*) $\det(A) \neq 0$.

(*j*) *$\lambda = 0$ is not an eigenvalue of A.*

(*k*) *T_A is one-to-one.*

(*l*) *T_A is onto.*