6.4 Composition and invertibility of linear transformations

Compositions of linear transformations

- A composition of functions: f:X->Y,g:Y->Z, we obtain g•f:X->Z.
- If f and g are linear, then g•f is also linear.
- To verify, we need to show + and scalar multiplications are preserved.

Theorem 6.4.1 *If $T_1 : R^n \to R^k$ and $T_2 : R^k \to R^m$ are both linear transformations, then $(T_2 \circ T_1) : R^n \to R^m$ is also a linear transformation.*

- Recall $T(x)=[T]x$ (p.272. (14))
- $(T_2 \bullet T_1)(e_i)= T_2(T_1(e_i))=[T_2]([T_1](e_i))=([T_2][T_1])(e_i)$. (final step why?)
- Thus $[T_2 \bullet T_1]=[T_2][T_1]$. (why?)
- Conversely, given matrices A and B,
- $T_B \bullet T_A = T_{BA}$. (Let $T_2=T_B, T_1=T_A$).
- Example 1. $R_\theta \bullet R_\Phi=R_{(\theta+\Phi)}$. Verify using computations
- Example 2. $H_\theta \bullet H_\Phi=R_{2(\Phi-\theta)}$.
- Example 3. $T \bullet S$ may not equal $S \bullet T$. We can see that from matrices $T_A \bullet T_B=T_{AB}$. $T_B \bullet T_A=T_{BA}$. They would be equal iff $AB=BA$.

Compositions of three or more linear transformations.

- $T\_1:R^n$->R^m,$T\_2:R^m$->R^l,$T\_3:R^l$->R^k We define T_3•T_2•T_1:R^n->R^k by
- T_3•T_2•T_1(x)=T_3(T_2(T_1(x))).
- Since the compositions are associative, we have (T_3•T_2)•T_1=T_3•(T_2•T_1). Thus we can drop the paranthese.
- [T_3•T_2•T_1]=[T_3][T_2][T_1].
  - [T_3•(T_2•T_1)]=[T_3][T_2•T_1]=[T_3]([T_2][T_1]).
  - We use matrix multiplications are associative.
- T_C•T_B•T_A=T_CBA

- A classification:
  - A rotation in R^3 <-> det A = 1.
  - A reflection composed with a rotation in R^3 <-> det A = -1.
- A product of series of rotations is a rotation.
- A product of series of reflections and rotations with an even number of reflections is a rotation.
- A product of series of reflections and rotations with an odd number of reflections is a reflection composed with a rotation.

Yaw, pitch and roll

- Yaw: z-axis (up direction), pitch:x-axis (wing direction), roll: y-axis (the direction of travel)
- Corresponding rotations are $R\_z\alpha$,$R\_y\beta$,$R\_x\gamma$.
- A composition of $R\_z\alpha$,$R\_y\beta$,$R\_x\gamma$ can be achieved by a single rotation $R\_v\delta$ in some direction of certain angle.
- Given these, we multiply them to get $R\_v\delta$, and then find the axis direction v and the rotation δ (between 0 and π).
- See Example 5.
- Conversely, any rotation can be factored into yaw, pitch, roll rotations.

Factoring linear operators ito compositions

- We wish to factor a matrix into elementary pieces so that we can understand it better.
- For example, a diagonal operator can be understood as a composition of contraction and expansion along individual axis. E
- We restrict to R^2 only.
- Example 7: There are five types of elementary matrices:

- (I) [[1,k],[0,1]] a shear in x-direction,
- (II) [[1,0],[k,1]] a shear in y-direction,
- (III) [[0,1],[1,0]] a reflection about x=y,
- (IV) [[k,0],[0,1]] compression or expansion for k≥0.
- (V) [[1,0],[0,k]] same. For k < 0, they are compression or expansion followed by a reflection.

Theorem 6.4.4 *If A is an invertible* 2×2 *matrix, then the corresponding linear operator on* R^2 *is a composition of shears, compressions, and expansions in the directions of the coordinate axes, and reflections about the coordinate axes and about the line* $y = x$.

- Example 8: illustrates the factorization and how one can understand a linear transformation.

Inverse

- $T: R^n \to R^m$. Suppose it is one-to-one.
- Let w be in the range of T.
- Then there is a unque x in R^n s.t. T(x)=w.
- Let $T^{-1}(w)$ be defined as x.
- $w=T(x) \leftrightarrow x=T^{-1}(w)$ for w in range(T).
- T^{-1}: range(T) -> R^n.
- TT^{-1} = Id on range (T)
- $T^{-1}T$=Id on R^n.

Theorem 6.4.5 *If T is a one-to-one linear transformation, then so is T^{-1}.*

Invertible linear operator

- If T is one-to-one and onto, then T^{-1} exists on the codomain, and is linear and one-to-one and onto. (The linearity already shown above. Other is just from the function theory)
- The matrix of T^{-1} is the inverse of the matrix of T.
  - $T^{-1}T(x)=[T^{-1}][T]x=x$. $[T^{-1}][T]=I$.

Theorem 6.4.6 *If T is a one-to-one linear operator on R^n, then the standard matrix for T is invertible and its inverse is the standard matrix for T^{-1}.*

- $[T^{-1}]=[T]^{-1}$.
- $(T\_A)^{-1}=T\_(A^{-1})$.
- An inverse of a rotation in R^2 is a rotation with opposite angle.
- An inverse of a rotation in R^3 is a rotation with the same axis with an opposite angle or an opposite axis with the same angle.
- An inverse of an expansion by k in an axis direction is a contraction by 1/k in the same axis direction.
- An inverse of a reflection is the same reflection. $H\_\theta H\_\theta = I$.

Inverse and linear system

- y=Ax given by a linear system as in (18).
- We have $x=A^{-1}y$ given by a linear system.
- We can obtain the second linear system by the first one by solving.
- Example 12.

Geometric properties of the invertible linear operators in R^2.

- What happens to lines, segments, polygons after acting by T?

Theorem 6.4.7 *If $T : R^2 \to R^2$ is an invertible linear operator, then:*

(*a*) *The image of a line is a line.*

(*b*) *The image of a line passes through the origin if and only if the original line passes through the origin.*

(*c*) *The images of two lines are parallel if and only if the original lines are parallel.*

(*d*) *The images of three points lie on a line if and only if the original points lie on a line.*

(*e*) *The image of the line segment joining two points is the line segment joining the images of those points.*

Theorem 6.4.8 *If $T : R^2 \to R^2$ is an invertible linear operator, then T maps the unit square into a nondegenerate parallelogram that has a vertex at the origin and has adjacent sides $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$. The area of this parallelogram is $|\det(A)|$, where $A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$ is the standard matrix for T.*