7.2. Properties of Basis

Unique linear combinations

- If v_1,v_2,v_3 is dependent, then one can write a given vector v in more than one way as a linear combinations.
- Example (1,1),(2,1),(3,2). (6,4)=2(3,2)=(1,1)+(2,1)+(3,2)
- For independent set, this does not happen: If there are two linear combinations, then we substract to get 0 vector written as a linear combination.
- This gives us "coordinates".

Theorem 7.2.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for subspace V of R^n, then every vector $\mathbf{v}$ in V can be expressed in exactly one way as a linear combination of the vectors in S.*

Removing and adding to get basis.

Theorem 7.2.2 *Let S be a finite set of vectors in a nonzero subspace V of R^n.*

(a) *If S spans V, but is not a basis for V, then a basis for V can be obtained by removing appropriate vectors from S.*

(b) *If S is a linearly independent set, but is not a basis for V, then a basis for V can be obtained by adding appropriate vectors from V to S.*

- Proof: (a) Choose the second vector. If it is a linear combination of the previous vectors, then remove it. Otherwise, leave it alone. The result still span V. Continue to the next one until we cannot remove any. The result is independent.
- (b) Omit.

Theorem 7.2.3 *If V is a nonzero subspace of R^n, then $\dim(V)$ is the maximum number of linearly independent vectors in V.*

- Proof: If {v_1,…,v_s} is independent with s maximum, then one cannot add any more vectors in V not in the span. This means V is the span of {v_1,…,v_s}.

Subspaces of subspaces: dimensions.

- V subspace of W, a subspace of R^n.
- What can we say about dimensions?
- This follows since if there is a basis in V, we can add vectors in W, to obtain a basis of W. dimV ≤dim W
- If the dimension is the same, then the basis of V is the basis of W and V=W.

Theorem 7.2.4 *If V and W are subspaces of R^n, and if V is a subspace of W, then:*

(a) $0 \leq \dim(V) \leq \dim(W) \leq n$

(b) $V = W$ *if and only if* $\dim(V) = \dim(W)$

- Some consequences

Theorem 7.2.5 *Let S be a nonempty set of vectors in R^n, and let S' be a set that results by adding additional vectors in R^n to S.*

(a) *If the additional vectors are in* $\text{span}(S)$, *then* $\text{span}(S') = \text{span}(S)$.

(b) *If* $\text{span}(S') = \text{span}(S)$, *then the additional vectors are in* $\text{span}(S)$.

(c) *If* $\text{span}(S')$ *and* $\text{span}(S)$ *have the same dimension, then the additional vectors are in* $\text{span}(S)$ *and* $\text{span}(S') = \text{span}(S)$.

Theorem 7.2.6

(a) *A set of k linearly independent vectors in a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(b) *A set of k vectors that span a nonzero k-dimensional subspace of R^n is a basis for that subspace.*

(c) *A set of fewer than k vectors in a nonzero k-dimensional subspace of R^n cannot span that subspace.*

(d) *A set with more than k vectors in a nonzero k-dimensional subspace of R^n is linearly dependent.*

Example

- v=(1,1,1),w=(2,1,-1),u=(2,0,1)
- Basis of R^3.
- Write (5,2,1) as a linear combination of v,w,u.

Theorem 7.2.7 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(a) The reduced row echelon form of A is I_n.

(b) A is expressible as a product of elementary matrices.

(c) A is invertible.

(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.

(g) $\det(A) \neq 0$.

(h) $\lambda = 0$ is not an eigenvalue of A.

(i) T_A is one-to-one.

(j) T_A is onto.

(k) The column vectors of A are linearly independent.

(l) The row vectors of A are linearly independent.

(m) The column vectors of A span R^n.

(n) The row vectors of A span R^n.

(o) The column vectors of A form a basis for R^n.

(p) The row vectors of A form a basis for R^n.