7_3 The fundamental spaces of a matrix.

Row space, column space, null space

- A mxn matrix
  - Row space of A: row(A): span of row vectors in R^n.
  - Column space of A: col(A): span of column vectors in R^m.
  - Null space of A: null(A): the solution space of Ax=O.
- In addition: $row(A^T)$, $col(A^T)$, $null(A^T)$
- $row(A^T) = col(A)$, $col(A^T) = row(A)$.
- So, row(A), col(A), null(A), $null(A^T)$ are fundamental spaces of A.

Definition 7.3.1 The dimension of the row space of a matrix A is called the ***rank*** of A and is denoted by rank(A); and the dimension of the null space of A is called the ***nullity*** of A and is denoted by nullity(A).

Orthogonal complements

Definition 7.3.2 If S is a nonempty set in R^n, then the ***orthogonal complement*** of S, denoted by $S^\perp$, is defined to be the set of all vectors in R^n that are orthogonal to every vector in S.

- Example: A is nxn-matrix. The solution space of Ax=0 is exactly the orthogonal complement of row vectors of A.
- Example: two vectors in R^3. The cross product solution.

Theorem 7.3.3 *If S is a nonempty set in R^n, then $S^\perp$ is a subspace of R^n.*

Properties of the orthogonal complements

Theorem 7.3.4

(*a*) *If W is a subspace of R^n, then* $W^\perp \cap W = \{\mathbf{0}\}$.

(*b*) *If S is a nonempty subset of R^n, then* $S^\perp = \text{span}(S)^\perp$.

(*c*) *If W is a subspace of R^n, then* $(W^\perp)^\perp = W$.

- Proof: (a) If v is in W^c and in W, then v is orthogonal to itself. v.v.=$||v||^2$=0. The length of v is zero and v is zero.
- (b) S^c is in span(S)^c since any vector v orthogonal to S is orthogonal to every vector in span(S).
  - Span(S)^c is in S^c. If v is orthogonal to Span(S), then v is orthogonal to S.
- (c) later.

row(A)^c=null(A)

Theorem 7.3.5 *If A is an $m \times n$ matrix, then the row space of A and the null space of A are orthogonal complements.*

Theorem 7.3.6 *If A is an $m \times n$ matrix, then the column space of A and the null space of A^T are orthogonal complements.*

- Proof: The solution space is a set of vectors orthogonal to the row vectors of A.
- row(A)^c=null(A), null(A)^c=row(A). (In R^n)
- col(A)^c=null(A^T), null(A^T)^c=col(A). (In R^m)

Theorem 7.3.7

(*a*) *Elementary row operations do not change the row space of a matrix.*

(*b*) *Elementary row operations do not change the null space of a matrix.*

(*c*) *The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of the matrix.*

The row operations will change the column space.

Theorem 7.3.8 *If A and B are matrices with the same number of columns, then the following statements are equivalent.*

(*a*) *A and B have the same row space.*

(*b*) *A and B have the same null space.*

(*c*) *The row vectors of A are linear combinations of the row vectors of B, and conversely.*

(a)<->(b). The null space is the orthogonal complement of the row space.

(c)->(a). Clear. (a)->(c). Row vectors of A span row space of B and conversely.

Finding basis by row reductions.

- S ={v_1,v_2,…,v_s}. Find a basis of Span S.
- 1. We form A where v_is are rows. Apply Gauss-Jordan elimination. This does not change the span and finds the basis.
- 2. Find a basis in S. This is slightly different. We will do this later.
- Example 4. Given four vectors in R^5, we use Gauss-Jordan elimination to obtain the echelon form. The basis is the set of row vectors.

- Example 4(b). Find a basis of W^c.
  - Form 4x5-matrix A. Obtain ref. Find the solution space and find its basis using the fundamental vectors.
- Example 5. Given v_1,v_2,v_3,v_4 in R^5, we find B such that the solution space of Bx=0 is span W.
  - Use the basis of W^c.

Determining whether a vector is in a given subspace.

- Problem 1. Given S={v_1,v_2,..,v_s} in R^m, determine a condition on b_1,…,b_m so that b=(b_1,…,b_m) will lie in span S.
- Problem 2. Given an mxn matrix A, find a condition on b_1,..,b_m so that b lies in col(A).
- Problem 3. Given a linear transformation T:R^n->R^m, determine a condition on b s.t. b is in ranT.
- Example 6.