7_4 The dimension theorem and its applications

Rank+nullity=dimension

The dimension theorem for matrices

- Let A be an mxn matrix.
- Ax=0. Let R be the ref of A.
- r nonzero rows, n-r free variables.
- r is the rank of R and hence that of A.
- nullity A = nullity R = n-r
- rank A+ nullity A = n=the number of columns

Theorem 7.4.1 (***The Dimension Theorem for Matrices***) *If A is an $m \times n$ matrix, then*

$$\text{rank}(A) + \text{nullity}(A) = n \quad (2)$$

- Example 1.

Expanding a linearly independent set to a basis

- {v_1,v_2,..,v_k} linearly independent in R^n.
- We can expend it to a basis.
  - First let A be the matrix with rows v_i.
  - The rank of A = k (Why?)
  - Solve Ax=0. The nullity A = n-k.
  - Find the basis of the solution space w_k+1, …, w_n.
  - v_is and w_js are orthogonal.
  - {v_1,…,v_k,w_k+1,..,w_n} are linearly independent and hence is a basis.
  - Example 2: read yourself.

Some consequences

- This is a useful theorem. (See Example 3,4)

Theorem 7.4.2 *If an $m \times n$ matrix A has rank k, then:*

(a) *A has nullity $n - k$.*

(b) *Every row echelon form of A has k nonzero rows.*

(c) *Every row echelon form of A has $m - k$ zero rows.*

(d) *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has k pivot variables (leading variables) and $n - k$ free variables.*

Theorem 7.4.3 ***(The Dimension Theorem for Subspaces)*** *If W is a subspace of R^n, then*

$$\dim(W) + \dim(W^{\perp}) = n \tag{3}$$

- Proof: If W={O}, trivially true.
  Suppose that W is not {O}.
  - Form a matrix A with rows the basis of W.
  - A is an mxn matrix. n is the dimension of R^n.
  - The row space of A is W.
  - The null space of A is W^c.
  - $\dim(W)+\dim(W^c)=\text{rank } A + \text{nullity } A = n$.

Theorem 7.4.4 *If A is an $n \times n$ matrix, and if T_A is the linear operator on R^n with standard matrix A, then the following statements are equivalent.*

(a) *The reduced row echelon form of A is I_n.*

(b) *A is expressible as a product of elementary matrices.*

(c) *A is invertible.*

(d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in R^n.*

(f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in R^n.*

(g) $\det(A) \neq 0$.

(h) *$\lambda = 0$ is not an eigenvalue of A.*

(i) *T_A is one-to-one.*

(j) *T_A is onto.*

(k) *The column vectors of A are linearly independent.*

(l) *The row vectors of A are linearly independent.*

(m) *The column vectors of A span R^n.*

(n) *The row vectors of A span R^n.*

(o) *The column vectors of A form a basis for R^n.*

(p) *The row vectors of A form a basis for R^n.*

(q) $\text{rank}(A) = n$.

(r) $\text{nullity}(A) = 0$.

Hyperplanes

Theorem 7.4.5 *If W is a subspace of R^n with dimension $n-1$, then there is a nonzero vector* $\mathbf{a}$ *for which $W = \mathbf{a}^{\perp}$; that is, W is a hyperplane through the origin of R^n.*

Theorem 7.4.6 *The orthogonal complement of a hyperplane through the origin of R^n is a line through the origin of R^n, and the orthogonal complement of a line through the origin of R^n is a hyperplane through the origin of R^n. Specifically, if* $\mathbf{a}$ *is a nonzero vector in R^n, then the line* $\text{span}\{\mathbf{a}\}$ *and the hyperplane* $\mathbf{a}^{\perp}$ *are orthogonal complements of one another.*

Rank 1 matrices: classification

- If A is of rank 1, then nullity A=n-1.
  - The row space of A is a line through O.
  - The null space of A is a hyperplane.
  - The converse also holds.
- If rank A =1, then row space of A is spanned by a single vector a.
  - Each row vector is a scalar multiple of a.
  - The null space A is a^c.
  - The converse also holds.

- How to obtain a rank 1 matrix. One take a vector v and multiply by scalars u_1, u_2,..,u_m and obtain u_1v, u_2v,..,u_mv. Take A to be the mxn matrix with these rows.
- Then A = uv^T for u=(u_1,u_2,...,u_m).
- Conversely, given a rank 1 matrix A, the rows of A are scalar multiple of some vector v. Listing the scalar multiples we form a vector u=(u_1,u_2,...,u_m).
- We obtain A= uv^T (See Example 8)

Theorem 7.4.7 *If* $\mathbf{u}$ *is a nonzero* $m \times 1$ *matrix and* $\mathbf{v}$ *is a nonzero* $n \times 1$ *matrix, then the outer product*

$$A = \mathbf{u}\mathbf{v}^T$$

has rank 1. *Conversely, if* A *is an* $m \times n$ *matrix with rank* 1, *then* A *can be factored into a product of the above form.*

Symmetric rank 1 matrices

- $A=uu^T$ is symmetric. ($A^T=uu^T$ also.)

Theorem 7.4.8 *If* $\mathbf{u}$ *is a nonzero* $n \times 1$ *column vector, then the outer product* $\mathbf{u}\mathbf{u}^T$ *is a symmetric matrix of rank* 1. *Conversely, if* A *is a symmetric* $n \times n$ *matrix of rank* 1, *then it can be factored as* $A = \mathbf{u}\mathbf{u}^T$ *or else as* $A = -\mathbf{u}\mathbf{u}^T$ *for some nonzero* $n \times 1$ *column vector* $\mathbf{u}$.