7.6 The pivot theorem

Basis problem

- Now address the problem of extracting a basis in S for the Span(S).
- The row operations changes the column spaces.
- If A and B are row equivalent, then Ax=0, Bx=0 have the same set of solutions.
- Ax=0 <-> x_1a_1+x_2a_2+…+x_na_n=0.
- Bx=0 <-> x_1b_1+x_2b_2+…+x_nb_n=0.

Theorem 7.6.1 *Let A and B be row equivalent matrices.*

(a) *If some subset of column vectors from A is linearly independent, then the corresponding column vectors from B are linearly independent, and conversely.*

(b) *If some subset of column vectors from B is linearly dependent, then the corresponding column vectors from A are linearly dependent, and conversely. Moreover, the column vectors in the two matrices have the same dependency relationships.*

- Proof: If necessary form A’ from the set of column vectors of A.
- Thus our strategy is to ref A and choose the pivot columns as basis and transfer back to A.
- Example 1.

Pivot theorem

Definition 7.6.2 The column vectors of a matrix A that lie in the column positions where the leading 1's occur in the row echelon forms of A are called the ***pivot columns*** of A.

Theorem 7.6.3 (***The Pivot Theorem***) *The pivot columns of a nonzero matrix A form a basis for the column space of A.*

- Proof: We see that leading 1s are at every position in the pivot column vectors.

Pivot algorithm

Algorithm 1 If W is the subspace of R^n spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, then the following procedure extracts a basis for W from S and expresses the vectors of S that are not in the basis as linear combinations of the basis vectors.

Step 1. Form the matrix A that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ as successive column vectors.

Step 2. Reduce A to a row echelon form U, and identify the columns with the leading 1's to determine the pivot columns of A.

Step 3. Extract the pivot columns of A to obtain a basis for W. If appropriate, rewrite these basis vectors in comma-delimited form.

Step 4. If it is desired to express the vectors of S that are not in the basis as linear combinations of the basis vectors, then continue reducing U to obtain the reduced row echelon form R of A.

Step 5. By inspection, express each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that contain leading 1's. Replace the column vectors in these linear combinations by the corresponding column vectors of A to obtain equations that express the column vectors of A that are not in the basis as linear combinations of basis vectors.

Example 2

- Given W=span(S). S finite.
- (a) Extract basis in S.
- (b) Express other vectors in S

Basis for the fundamental spaces

- A mxn -> U upper echelon -> R ref.
- 1. row(A): basis nonzero rows of U or R.
- 2. col(A): pivot columns of A.
- 3. null(A): canonical solutions from Rx=0.
- 4. null(A^T): Solve $A^Tx=0$.
- A mxn rank k. dim null(A^T)= m-k. Why? If k=m, dim=0.
- Another method using row operations only.

Algorithm 2 If A is an $m \times n$ matrix with rank k, and if $k < m$, then the following procedure produces a basis for null(A^T) by elementary row operations on A.

Step 1. Adjoin the $m \times m$ identity matrix I_m to the right side of A to create the partitioned matrix $[A \mid I_m]$.

Step 2. Apply elementary row operations to $[A \mid I_m]$ until A is reduced to a row echelon form U, and let the resulting partitioned matrix be $[U \mid E]$.

Step 3. Repartition $[U \mid E]$ by adding a horizontal rule to split off the zero rows of U. This yields a matrix of the form

$$\left[\begin{array}{c|c} V & E_1 \\ \hline 0 & E_2 \end{array}\right] \begin{matrix} k \\ m-k \end{matrix}$$
$$\begin{matrix} n & m \end{matrix}$$

where the margin entries indicate sizes.

Step 4. The row vectors of E_2 form a basis for null(A^T).

- Example 3:

Column-row factorization

Theorem 7.6.4 (*Column-Row Factorization*) *If A is a nonzero $m \times n$ matrix of rank k, then A can be factored as*

$$A = CR \tag{1}$$

where C is the $m \times k$ matrix whose column vectors are the pivot columns of A and R is the $k \times n$ matrix whose row vectors are the nonzero rows in the reduced row echelon form of A.

- Proof: EA=R_0. E mxm matrix a product of elementary matrices.
  - R_0 ref of A. mxn-matrix
  - Let R be the kxn-matrix of nonzero rows of R_0.
  - Then let E^{-1}=[C|D] C mxk. D mx(m-k)

$$R_0 = \left[\frac{R}{O}\right]$$

- Proof continued:
- $A=E^{-1}R=$

$$[C \mid D]\left[\frac{R}{O}\right] = CR + DO = CR$$

- C consists of pivot columns of A.
  - Multiplying by E^{-1} to R_0 returns to A.
  - Restrict to pivot columns of R -> pivot columns of A.
  - Pivot columns of R form I of kxk size.
  - CR restricted CI=C. Thus C is the pivot columns of A.
- Example 4.

Column-row expansion

- We can write the above as the sum of vector products…

Theorem 7.6.5 (***Column-Row Expansion***) *If A is a nonzero matrix of rank k, then A can be expressed as*

$$A = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdots + \mathbf{c}_k\mathbf{r}_k \tag{4}$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k$ are the successive pivot columns of A and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ are the successive nonzero row vectors in the reduced row echelon form of A.

Column-row rule (Theorem 3.8.1)

$$A^{m\times s} = [c_1, c_2, \ldots, c_s], B^{sxn} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_s \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^{s} A_{ik} B_{kj} = \sum_{k=1}^{s} c_{ik} r_{kj}$$

$$AB = \sum_{k=1}^{s} c_k r_k = c_1 r_1 + c_2 r_2 + \cdots c_s r_s$$