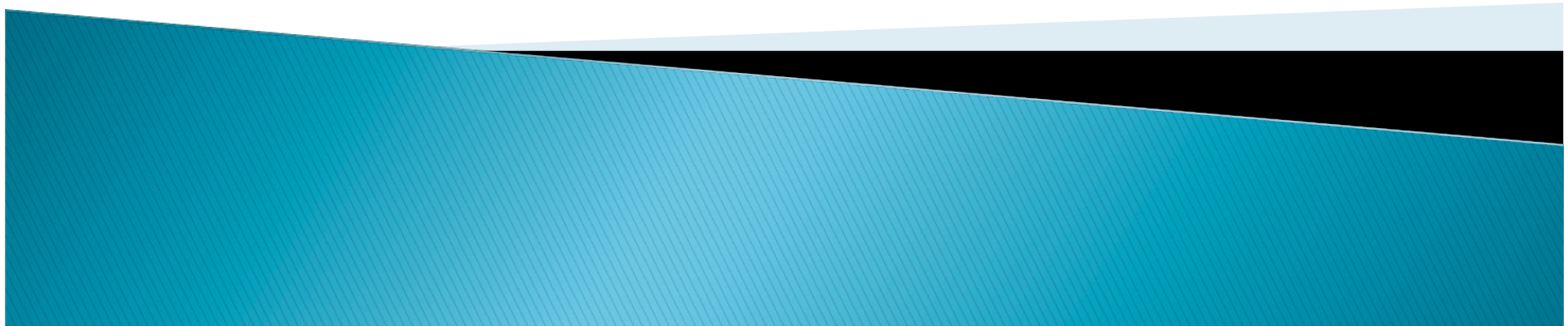


3.3. Elementary matrices

A method to find A^{-1}



Elementary matrices

- ▶ Elementary operations:
 - 1. Interchange two rows.
 - 2. Multiply a row by a nonzero constant
 - 3. Add a multiple of one row to another.
- ▶ Elementary matrix is a matrix that results from a single elementary row operation to I_n .

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem 3.3.1 *If A is an $m \times n$ matrix, and if the elementary matrix E results by performing a certain row operation on the $m \times m$ identity matrix, then the product EA is the matrix that results when the same row operation is performed on A .*

- ▶ See Example 1.
- ▶ Each elementary operation has an elementary operation that reverses it.
 - 1) multiply a row by c (nonzero)
 - 1') multiply the same row by $1/c$
 - 2) interchange row i and row j . 2')=2)
 - 3) Add c times a row i to row j .
 - 3') Add $-c$ times a row i to row j .



- ▶ An elementary matrix $E=e(I)$ for an operation e . Then $E'=e'(I)$ is the inverse of E .
- ▶ Proof: $EE'=e(E')=e(e'(I)) = I$.
 $E'E=e'(E)=e'(e(I))=I$.

Theorem 3.3.2 *An elementary matrix is invertible, and the inverse is also an elementary matrix.*



Theorem 3.3.3 *If A is an $n \times n$ matrix, then the following statements are equivalent; that is, they are all true or all false.*

- (a) *The reduced row echelon form of A is I_n .*
- (b) *A is expressible as a product of elementary matrices.*
- (c) *A is invertible.*

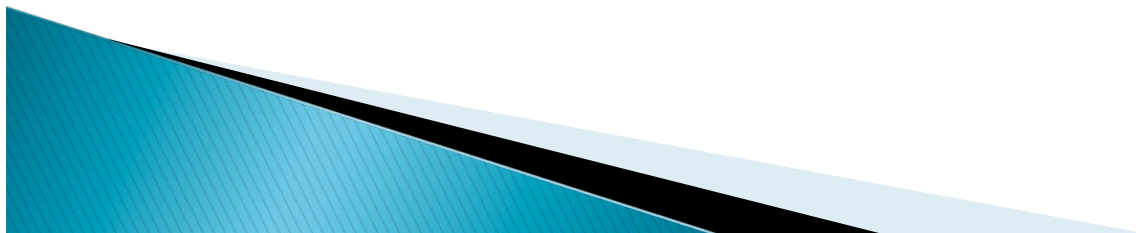
▶ **Proof:** (a) \rightarrow (b) \rightarrow (c) \rightarrow (a)

▶ (a) \rightarrow (b).

- Ref of A is I_n . There is a sequence of elementary moves making A into I_n . Each operation is a multiplication by an elementary matrix.
- $E_k E_{k-1} \dots E_2 E_1 A = I_n$.
- $A = E_1^{-1} E_2^{-1} \dots E_{k-1}^{-1} E_k^{-1} I_n$



- ▶ (b) \rightarrow (c)
 - If A is a product of elementary matrices and each elementary matrix is invertible. Thus, so is A .
- ▶ (c) \rightarrow (a) Suppose A is invertible.
 - Let R be ref of A .
 - Then $E_k \dots E_2 E_1 A = R$.
 - Then R is invertible. By Theorem 3.2.4, either R has zero rows or $R = I_n$. The former implies R is not invertible.
 - Thus $R = I_n$.



Row equivalence

- ▶ If a matrix B is obtained from A by applying a sequence of row operations. Then B is row equivalent to A . $A \cong B$.
- ▶ $A \cong B$ iff $B \cong A$, $A \cong B$, $B \cong C \rightarrow A \cong C$, $A \cong A$.

A square matrix A is invertible if and only if it is row equivalent to the identity matrix of the same size.

Theorem 3.3.4 *If A and B are square matrices of the same size, then the following are equivalent:*

- A and B are row equivalent.*
- There is an invertible matrix E such that $B = EA$.*
- There is an invertible matrix F such that $A = FB$.*

Inversion algorithm

The Inversion Algorithm *To find the inverse of an invertible matrix A , find a sequence of elementary row operations that reduces A to I , and then perform the same sequence of operations on I to obtain A^{-1} .*

- ▶ Proof: $E_k \dots E_2 E_1 A = I$. $E_k \dots E_2 E_1 I = A^{-1}$
- ▶ Example 3.



Solving linear equations by matrix inversions

- ▶ If A is invertible, then $Ax=b$ can be solved by $x = A^{-1}b$.

Theorem 3.3.5 *If $Ax = b$ is a linear system of n equations in n unknowns, and if the coefficient matrix A is invertible, then the system has a unique solution, namely $x = A^{-1}b$.*

Theorem 3.3.6 *If $Ax = 0$ is a homogeneous linear system of n equations in n unknowns, then the system has only the trivial solution if and only if the coefficient matrix A is invertible.*

- ▶ Proof) \Leftarrow) $Ax=0$. $x=A^{-1}0=0$.
- ▶ \rightarrow) Let A' be A augmented with 0 column.
- ▶ Then the ref for the augmented A' is I augmented with 0s since 0s are the only solutions.
- ▶ augmented. Thus ref of $A=I$.



Theorem 3.3.8

- (a) *If A and B are square matrices such that $AB = I$ or $BA = I$, then A and B are both invertible, and each is the inverse of the other.*
- (b) *If A and B are square matrices whose product AB is invertible, then A and B are invertible.*

- ▶ **Proof:** (a) Suppose $AB=I$.
- Then show $Bx=0$ has a unique solution 0 . Use Thm 3.3.6.
 - $ABx=0 \rightarrow Ix=0$. Thus $x=0$. Done. B is invertible.
 - $ABB^{-1}=B^{-1}$. Thus $A=B^{-1}$ and A is invertible.
- ▶ (b) AB invertible. $I=(AB)(AB)^{-1}=A(B(AB)^{-1})$.
 $I=(AB)^{-1}(AB)=((AB)^{-1}A)B$.

Theorem 3.3.9 *If A is an $n \times n$ matrix, then the following statements are equivalent.*

- (a) *The reduced row echelon form of A is I_n .*
- (b) *A is expressible as a product of elementary matrices.*
- (c) *A is invertible.*
- (d) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
- (e) *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector \mathbf{b} in R^n .*
- (f) *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector \mathbf{b} in R^n .*

- ▶ Proof: (a)(b)(c)(d) equivalent.
- ▶ We show (c)(e)(f) equivalent. (f) \rightarrow (e) \rightarrow (c) \rightarrow (f).
- ▶ (f) \rightarrow (e). Has a solution \rightarrow consistent.
- ▶ (e) \rightarrow (c). $A\mathbf{x}=\mathbf{e}_1, A\mathbf{x}=\mathbf{e}_2, \dots, A\mathbf{x}=\mathbf{e}_n$ are consistent.
 - Let $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n$ be the respective solutions.
 - Then $A[\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n] = I_n$. $C = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n]$ is an $n \times n$ -matrix. By Theorem 3.3.8, A is invertible.
- ▶ (c) \rightarrow (f) Theorem 3.3.5.

- ▶ Solving multiple linear system with common coefficients.
 - We can simplify by stacking bs.
 - See Example 7.
- ▶ Consistency of linear systems

3.3.10 The Consistency Problem For a given matrix A , find all vectors \mathbf{b} for which the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

- ▶ Example 8.



Ex. Set. 3.3.

- ▶ 1-6 Recognizing elementary matrices, finding inverse. 7,8.
- ▶ 9-12 Find inverse
- ▶ 13-14,15-22 Inverse finding
- ▶ 29-34 Find consistency conditions.
- ▶ D5-D7... Interesting theoretical sides...

