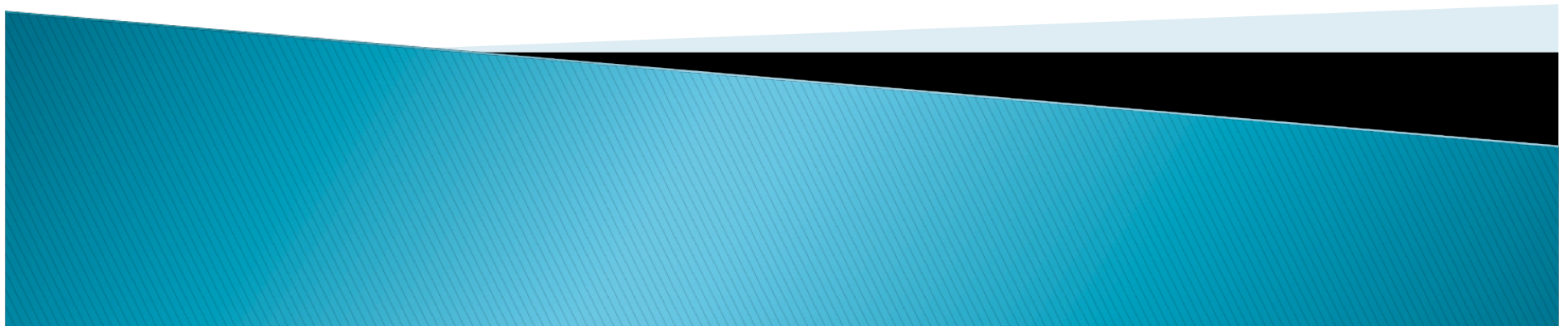


# 3.5. The geometry of linear systems

Solutions for inhomogeneous systems.

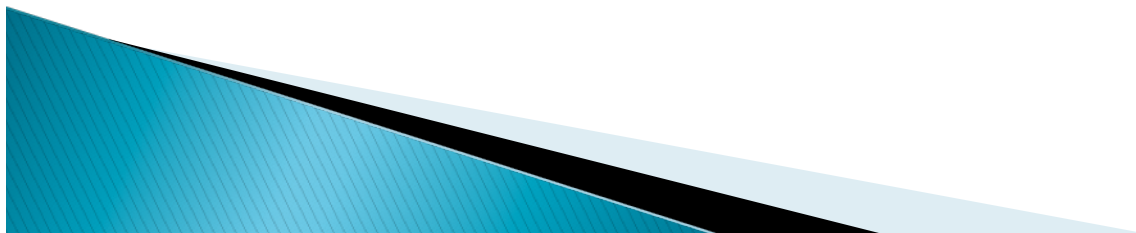
Consistency

Geometric interpretations



# Translated subspaces

- ▶  $W$  is a subspace.
- ▶  $x_0+W=\{v=x_0+w|w \text{ is in } W\}$
- ▶ This is not a subspace in general but is called an affine subspace (linear manifold, flat).
- ▶ For example  $x_0+\text{span}\{v_0,v_1,\dots,v_s\}$   
 $=\{v=x_0+c_0v_1+\dots+c_s v_s\}$
- ▶  $y=1$  in  $\mathbb{R}^2$ .  $\{(x,1)|x \text{ in } \mathbb{R}\}=(0,1)+\{(x,0)|x \text{ in } \mathbb{R}\}$
- ▶  $Ax+By+Cz=D$  in  $\mathbb{R}^3$  translated from  $Ax+By+Cz=0$  since they are parallel.



# The solution space of $Ax=b$ and that of $Ax=0$

- ▶  $W=\{x|Ax=b\}$ ,  $W_0=\{x|Ax=0\}$
- ▶ Let  $x$  be in  $W$ . Take one  $x_0$  in  $W$ . Then  $x-x_0$  is in  $W_0$ .
  - $A(x-x_0)=Ax-Ax_0=b-b=0$ .
- ▶ Given an element  $x$  in  $W_0$ .  $x+x_0$  is in  $W$ .
- ▶  $A(x+x_0)=Ax+Ax_0=0+b=b$ .
- ▶ Thus,  $W=x_0+W_0$ .

**Theorem 3.5.1** *If  $Ax = b$  is a consistent nonhomogeneous linear system, and if  $W$  is the solution space of the associated homogeneous system  $Ax = 0$ , then the solution set of  $Ax = b$  is the translated subspace  $x_0 + W$ , where  $x_0$  is any solution of the nonhomogeneous system  $Ax = b$  (Figure 3.5.1).*

- ▶  $W = \{(x,y) | x+y=1\}$  is obtained from
- ▶  $W_0 = \{(x,y) | x+y=0\}$  adding  $(1,0)$  in  $W$ .
- ▶  $W = \{(x,y,z) | Ax+By+Cz=D\}$  is obtained from  $W_0 = \{(x,y,z) | Ax+By+Cz=0\}$  by a translation by  $(x_0, y_0, z_0)$  for any point of  $W$ .
- ▶  $W = \{(x,y,z) | x+y+z=1, x-y=0\}$ 
  - $= \{(s+1/2, 1/2, s) | s \text{ in } \mathbb{R}\}$
  - $= \begin{bmatrix} s + 1/2 \\ 1/2 \\ s \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
  - Here  $(1/2, 1/2, 0)$  is in  $W$  and  $\{s(1,0,1)\}$  are solutions of the homogeneous system.



- ▶ Solution to  $A\mathbf{x}=\mathbf{b}$  can be written as  $\mathbf{x}=\mathbf{x}_h+\mathbf{x}_0$  where  $\mathbf{x}_0$  is a particular solution and  $\mathbf{x}_h$  is a homogeneous solution.

**Theorem 3.5.2** *A general solution of a consistent linear system  $A\mathbf{x} = \mathbf{b}$  can be obtained by adding a particular solution of  $A\mathbf{x} = \mathbf{b}$  to a general solution of  $A\mathbf{x} = \mathbf{0}$ .*

**Theorem 3.5.3** *If  $A$  is an  $m \times n$  matrix, then the following statements are equivalent.*

- (a)  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
- (b)  $A\mathbf{x} = \mathbf{b}$  has at most one solution for every  $\mathbf{b}$  in  $R^m$  (i.e., is inconsistent or has a unique solution).

**Theorem 3.5.4** *A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.*

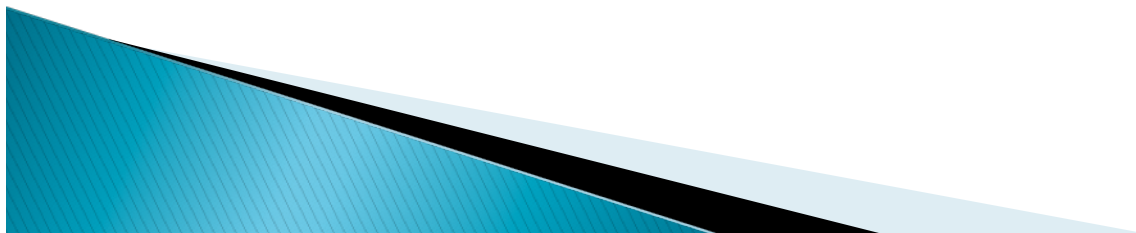


# Consistency of a linear equation.


- ▶  $Ax=b$  can be written as  
 $x_1v_1+x_2v_2+\dots+x_nv_n=b.$

**Theorem 3.5.5** *A linear system  $Ax = b$  is consistent if and only if  $b$  is in the column space of  $A$ .*

- ▶ This can be used to tell whether a certain vector can be written as a linear combination of some other vectors
- ▶ Example 2.



# Hyperplanes

- ▶  $a_1x_1+a_2x_2+\dots+a_nx_n=b$  in  $\mathbb{R}^n$ . ( $a_i$  not all zero)
  - ▶ The set of points  $(x_1,x_2,\dots,x_n)$  satisfying the equation is said to be a hyperplane.
  - ▶  $b=0$  if and only if the hyperplane passes  $O$ .
  - ▶ We can rewrite  $a.x=b$  where  $a=(a_1,\dots,a_n)$  and  $x=(x_1,\dots,x_n)$ .
  - ▶ A hyperplane with normal  $a$ .
  - ▶  $a.x=0$ . An orthogonal complement of  $a$ .
  - ▶ Example 3.
- 

# Geometric interpretations of solution spaces.

- ▶  $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
- ▶  $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$
- ▶ .....
- ▶  $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$
  
- ▶ This can be written:  $a_1 \cdot x = 0, a_2 \cdot x = 0, \dots, a_m \cdot x = 0.$





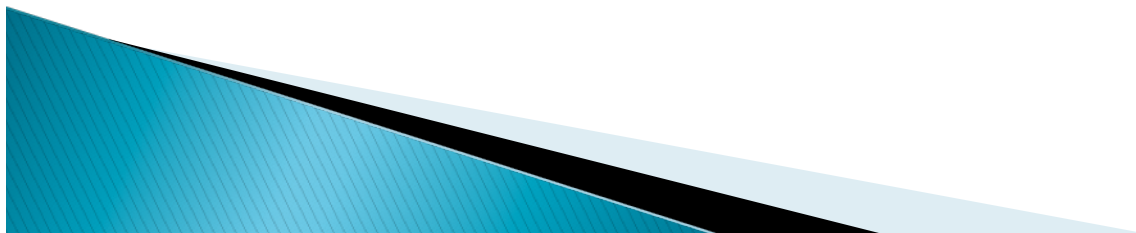
**Theorem 3.5.6** *If  $A$  is an  $m \times n$  matrix, then the solution space of the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  consists of all vectors in  $R^n$  that are orthogonal to every row vector of  $A$ .*

- ▶ See Example 4



# Look ahead

- ▶ The set of solutions of a system of linear equation can be solved by Gauss-Jordan method.
- ▶ The result is the set  $W$  of vectors of form  $x_0 + t_1 v_1 + \dots + t_s v_s$  where  $t_i$  are free variables.
- ▶ We show that  $\{v_1, v_2, \dots, v_n\}$  is linearly independent later.
- ▶ Thus  $W = x_0 + W_0$ .  $W$  is an affine subspace of dimension  $s$ .



# Ex. Set 3.5.

- ▶ 1-4 solving
- ▶ 5-8 linear combinations
- ▶ 7-10 span
- ▶ 11-20 orthogonality

