3.6. Matrices with special forms

Diagonal matrix, triangular matrix, symmetric and skew-symmetric matrices, AA^T, Fixed points, inverting I-A

Diagonal matrices

- A square matrix where non-diagonal entries are 0 is a diagonal matrix.
- d_1, d_2,… are real numbers (could be zero.) O, I diagonal matrices

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

- If every diagonal entry is not zero, then the matrix is invertible.
- The inverse is a diagonal matrix with diagonal entries $1/d_1$, $1/d_2$,…, $1/d_n$.
- D^k for positive integer k is diagonal with entries d_1^k,…,d_n^k.
- See Example 1.
- Left multiplication of the matrix by a diagonal matrix. Right multiplication of the matrix by a diagonal matrix.

Triangular matrices

- Given a square matrix.
- Lower triangular matrices: entries above the diagnonals $a_{ij} = 0$ if $i < j$.
- Upper triangular matrices:entries below the diagonals $a_{ij}=0$ if $i > j$.
- A lower triangular matrix or an upper triangular matrix are triangular.
- Row echelon forms are upper triangular.

Theorem 3.6.1

(*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(*b*) *A product of lower triangular matrices is lower triangular, and a product of upper triangular matrices is upper triangular.*

(*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(*d*) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

- Proof: (b) A,B both upper triangular.
  - (AB)_ij = 0 if i>j.

$$\begin{bmatrix} 0 & \cdots & 0 & a_{ii} & a_{i(i+1)} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  - (c),(d) proved later
- See Example 4

Symmetric and skew-symmetric matrices

- A square matrix A is symmetric if $A^T=A$ or A_ij=A_ji.
- A is skew-symmetric if $A^T=-A$ or A_ij=-A_ji.

Theorem 3.6.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(a) A^T *is symmetric.*

(b) $A+B$ *and* $A-B$ *are symmetric.*

(c) kA *is symmetric.*

Theorem 3.6.3 *The product of two symmetric matrices is symmetric if and only if the matrices commute.*

$(AB)^T=B^TA^T=BA$. This equals AB iff AB=BA iff A and B commute.

- A,B skew-symmetric $(AB)^T=B^TA^T=(-B)(-A)=BA = AB$ iff A and B commute.
  (AB is symmetric in fact.)
- The right conditions is BA=-AB (anticommute)

Invertible symmetric matrix.

- A symmetric matrix may not by invertible.
- Example: 2x2 matrix with all entries 1 is symmetric but not invertible.

Theorem 3.6.4 *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

- Proof: $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ as A is symmetric. Thus A^{-1} is symmetric also.

AA^T, A^TA (A need not be square.)

- AA^T is symmetric ($(AA^T)^T=(A^T)^TA^T=AA^T$.)
- Similary A^TA is symmetric.
- If row vectors of A are r_1,r_2,..,r_n, then the column vectors of A^T are r_1^T,r_2^T,…,r_n^T.

$$AA^T = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} r_1^T & r_2^T & \cdots & r_n^T \end{bmatrix} = \begin{bmatrix} r_1 r_1^T & r_1 r_2^T & \cdots & r_1 r_n^T \\ r_2 r_1^T & r_2 r_2^T & \cdots & r_2 r_n^T \\ \vdots & \vdots & \ddots & \vdots \\ r_n r_1^T & r_n r_2^T & \cdots & r_n r_n^T \end{bmatrix}$$

$$AA^T = \begin{bmatrix} r_1 . r_1 & r_1 . r_2 & \cdots & r_1 . r_n \\ r_2 . r_1 & r_2 . r_2 & \ldots & r_2 . r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_n . r_1 & r_n . r_2 & \ldots & r_n . r_n \end{bmatrix}$$

Theorem 3.6.5 *If A is a square matrix, then the matrices A, AA^T, and A^TA are either all invertible or all singular.*

If A is invertible, then so is A^T and hence AA^T and A^TA are invertible.
If A^TA or AA^T are invertible, the use 3.3.8 (b) to prove this.

I-A.

- A fixed point x of A: Ax=x.
- We find x by solving (I-A)x=0.
- Fixed points can be useful.
- Example 6.
- Finding the inverse of I-A are often useful in applications. Suppose $A^k=0$ for some positive k.

- Recall the polynomial algebra:
  - $(1-x)(1+x+\ldots+x^{k-1})=1-x^k$.
  - Plug A in to obtain $(I-A)(I+A+\ldots+A^{k-1})=I-A^k=I$.
  - Thus $(I-A)^{-1}=I+A+\ldots+A^{k-1}$.
  - Examples: Strictly upper triangular or strictly lower triangular matrices…
  - Those that are of form BAB^{-1} for A strictly triangular.

Using power series to obtain approximate inverse to I-A.

- For real x with $|x| < 1$, we have a formular $(1-x)^{-1}=1+x+x^2+\ldots+x^n+\ldots$
- This converges absolutely.
- We plug in A to obtain $(I-A)^{-1}=I+A+A^2+\ldots+A^n+\ldots$
- Again this will converge under the condition that sum of absolute values of each column (or each row) is less than 1.
- Basic reason $A^n \to O$ as $n \to \infty$.
- (see **Leontief Input-Output Economic Model**)

Ex Set 3.6.

- 1-6. Diagonal matrices
- 7-10 Triangular matrices
- 11-24 Symmetric matrices, inverse…
- 25,26 Inverse of I-A