Chapter 3:
Linear transformations

Linear transformations, Algebra of linear transformations, matrices, dual spaces, double duals

Linear transformations

- V, W vector spaces with same fields F.
  - Definition: T:V➔W s.t. T(ca+b)=c(Ta)+Tb for all a,b in V. c in F. Then T is linear.
  - Property: T(O)=O. T(ca+db)=cT(a)+dT(b), a,b in V, c,d in F. (equivalent to the def.)
  - Example: A mxn matrix over F. Define T by Y=AX. $T:F^n \rightarrow F^m$ is linear.
    - Proof: T(aX+bY)= A(aX+bY)=aAX+bAY = aT(X) +bT(Y).

- U:F^{1xm} ->F^{1xn} defined by U(a)=aA is linear.
- Notation: $F^m=F^{mx1}$ (not like the book)
- Remark: L(F^{mx1},F^{nx1}) is same as M_{mxn}(F).
  - For each mxn matrix A we define a unique linear transformation Tgiven by T(X)=AX.
  - For each a linear transformation T has A such that T(X)=AX. We will discuss this in section 3.3.
  - Actually the two spaces are isomorphic as vector spaces.
  - If m=n, then compositions correspond to matrix multiplications exactly.

- Example: T(x)=x+4. F=R. V=R. This is not linear.
- Example: V = {f polynomial:F➔F}
  T:V ➔V defined by T(f)=Df.

$$
\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k \\
Df(x) &= c_1 + 2c_2 x + \cdots + k c_k x^{k-1}
\end{aligned}
$$

- V={f:R➔R continuous}

$$Tf(x) = \int_0^x f(t)dt$$

- Theorem 1: V vector space over F. basis $\alpha_1, \ldots, \alpha_n$. W another one with vectors $\beta_1, ..., \beta_m$ (any kind m≥n). Then exists a unique linear tranformation T:V➔W s.t. $T(\alpha_j) = \beta_j, j = 1, \ldots, n$

- Proof: Check the following map is linear.

$$\begin{aligned} \alpha &= x_1\alpha_1 + \cdots + x_n\alpha_n \\ T\alpha &= x_1\beta_1 + \cdots + x_n\beta_n \end{aligned}$$

- Null space of T :V➔W:= { v in V| Tv = 0}.
- Rank T:= dim{Tv|v in V} in W. = dim range T.
- Null space is a vector subspace of V.
- Range T is a vector subspace of W.
- Example:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Null space z=t=0. x+2y=0 dim =1
- Range = W. dim = 3

- Theorem: rank T + nullity T = dim V.
- Proof: $a_1,..,a_k$ basis of N. dim N = k. Extend to a basis of V: $a_1,..,a_k, a_{k+1}, \ldots,a_n$.
  - We show $T a_{k+1},\ldots,Ta_n$ is a basis of R. Thus n-k = dim R. n-k+k=n.
    - Spans R:

$$\begin{aligned} v &= x_1\alpha_1 + \cdots + x_n\alpha_n \\ Tv &= x_{k+1}T(\alpha_{k+1}) + \cdots x_nT(\alpha_n) \end{aligned}$$

    - Independence:

$$\begin{aligned} \sum_{i=k+1}^{n} c_i T\alpha_i &= 0 \\ T(\sum_{i=k+1}^{n} c_i\alpha_i) &= 0 \\ \sum_{i=k+1}^{n} c_i\alpha_i &\in N \\ \sum_{i=k+1}^{n} c_i\alpha_i &= \sum_{i=1}^{k} c_i\alpha_i \\ c_i &= 0, i = k+1,\ldots,n \end{aligned}$$

- Theorem 3: A mxn matrix.
  Row rank A = Column rank A.
- Proof:
  - column rank A = rank T where $T:R^n \rightarrow R^m$ is defined by Y=AX. e_i goes to i-th column. So range is spaned by column vectors.
  - rankT+nullityT=n by above theorem.
  - column rank A+ dim S = n where S={X|AX=O} is the null space.
  - dim S= n - row rank A (Example 15 Ch. 2 p.42)
  - row rank = column rank.

- (Example 15 Ch. 2 p.42) A^{mxn}. S solution space. R r-r-e matrix
- r = number of nonzero rows of R.
- RX=0 $k_1<k_2<\ldots<k_r$. J= {1,..,n}- $\{k_1,k_2,\ldots,k_r\}$.

$$\begin{array}{lllllll}
x_{k_1} & & & & + & \sum_{j=1}^{n-r} C_{1j}u_j & = 0 \\
& x_{k_2} & & & + & \sum_{j=1}^{n-r} C_{2j}u_j & = 0 \\
& & \ddots & & + & \vdots & = \vdots \\
& & & x_{k_r} & + & \sum_{j=1}^{n-r} C_{rj}u_j & = 0
\end{array}$$

  - Solution spaces parameter $u_1,\ldots,u_{n-r}$.
  - Or basis E_j given by setting u_j =1 and other u_i= 0 and $x_{ki}= c_{ij}$.

Algebra of linear transformations

- Linear transformations can be added, and multiplied by scalars. Hence they form a vector space themselves.
- Theorem 4: $T,U:V \rightarrow W$ linear.
  - Define $T+U:V \rightarrow W$ by $(T+U)(a)=T(a)+U(a)$.
  - Define $cT:V \rightarrow W$ by $cT(a)=c(T(a))$.
  - Then they are linear transformations.

- Definition: L(V,W)={T:V➔W| T is linear}.
- Theorem 5: L(V,W) is a finite dim vector space if so are V,W. dimL=dimVdimW.
- Proof: We find a basis: $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \subset V$ $\mathcal{B}' = \{\beta_1, \ldots, \beta_m\} \subset W$
  - Define a linear transformation V➔W:

$$E^{p,q}(\alpha_i) = \begin{cases} 0, & i \neq q \\ \beta_p, & i = q \end{cases} = \delta_{iq}\beta_p, \quad 1 \leq p \leq m, 1 \leq q \leq n$$

  - The basis:

$$\begin{matrix} E^{1,1}, & \ldots, & E^{1,n} \\ \vdots & \ddots & \vdots \\ E^{m,1}, & \ldots, & E^{m,n} \end{matrix}$$

– Spans: T:V➔W.

$$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p$$

- We show

$$T = U = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{pq}E^{p,q}$$

$$U(\alpha_j) = \sum_{p=1}^{m}\sum_{q=1}^{n} A_{p,q}E^{p,q}(\alpha_j)$$

$$= \sum_{p=1}^{m}\left(\sum_{q=1}^{n} A_{p,q}\delta_{j,q}\right)(\beta_p)$$

$$= \sum_{p=1}^{m} A_{pj}\beta_p = T\alpha_j, j = 1,..,m$$

$$T = U$$

– Independence

• Suppose

$$
\begin{aligned}
U &= \sum_p \sum_q A_{pq} E^{p,q} = 0 \\
U\alpha_j &= 0 \\
\sum_p A_{pj}\beta_p &= 0 \\
\{\beta_p\} & \quad \text{independent} \\
A_{pj} &= 0 \text{ for all } p, j
\end{aligned}
$$

• Example: $V=F^m$ $W=F^n$. Then

– $M_{mxn}(F)$ is isomorphic to $L(F^m,F^n)$ as vector spaces. Both dimensions equal mn.

– $E^{p,q}$ is the mxn matrix with 1 at (p,q) and 0 everywhere else.

– Any matrix is a linear combination of $E^{p,q}$.

- Theorem. $T:V\rightarrow W$, $U:W\rightarrow Z$.
  $UT:V\rightarrow Z$ defined by $UT(a)= U(T(a))$ is linear.
- Definition: Linear operator $T:V\rightarrow V$.
- $L(V,V)$ has a multiplication.
  - Define $T^0=I$, $T^n=T\dots T$. n times.
  - Example: A mxn matrix B pxm matrix
    T defined by $T(X)=AX$. U defined by $U(Y)=BY$.
    Then $UT(X) = BAX$. Thus
    UT is defined by BA if T is defined by A and U by B.
  - Matrix multiplication is defined to mimic composition.

- Lemma:
  - $IU=UI=U$
  - $U(T_1+T_2)=UT_1+UT_{2,}$ $(T_1+T_2)U=T_1U+T_2U$.
  - $c(UT_1)= (cU)T_1=U(cT_1)$.
- Remark: This make $L(V,V)$ into linear algebra (i.e., vector space with multiplications) in fact same as the matrix algebra $M_{nxn}(F)$ if $V=F^n$ or more generally dim $V = n$. (Example 10. P.78)

- Example: V={f:F→F| f is a polynomial}.
  - D:V→V differentiation.

$$
\begin{aligned}
f(x) &= c_0 + c_1 x + \cdots + c_n x^n \\
Df(x) &= c_1 + \cdots + n c_n x^{n-1}
\end{aligned}
$$

  - T:V→V: T sends f(x) to xf(x)
  - DT-TD = I. We need to show DT-TD(f)= f for each polynomial f.
  - (QP-PQ=ihI In quantum mechanics.)

Invertible transformations

- $T:V \rightarrow W$ is invertible if there exists $U:W \rightarrow V$ such that $UT=I_v$, $TU=I_w$. U is denoted by T^{-1}.
- Theorem 7: If T is linear, then T^{-1} is linear.
- Definition: $T:V \rightarrow W$ is nonsingular if $Tc=0$ implies $c=0$
  - Equivalently the null space of T is {O}.
  - T is one to one.
- Theorem 8: T is nonsingular iff T carries each linearly independent set to a linearly independent set.

- Theorem 9: V, W dim V = dim W. T:V ➔W is linear. TFAE:
  - T is invertible.
  - T is nonsingular
  - T is onto.
- Proof: We use n=dim V = dim W. rank T+nullity T = n.
  - (ii) iff (iii): T is nonsingular iff nullity T =0 iff rank T =n iff T is onto.
  - (I)➔(ii): TX=0, T^{-1}TX=0, X=0.
  - (ii)➔(i): T is nonsingular. T is onto. T is 1-1 onto. The inverse function exists and is linear. T^{-1} exists.

Groups

- A group (G, .):
  - A set G and an operation GxG->G:
    - x(yz)=(xy)z
    - There exists e s.t. xe=ex=x
    - To each x, there exists x^{-1} s.t. xx^{-1}=e and $x^{-1}x$=e.
- Example: The set of all 1-1 maps of {1,2,…,n} to itself.
- Example: The set of nonsingular maps GL(V,V) form a group.

Isomorphisms

- V, W T:V->W one-to-one and onto (invertible). Then T is an isomorphism. V,W are isomorphic.
- Isomorphic relation is an equivalence relation: V~V, V~W <-> W~V, V~W, W~U -> V~W.

- Theorem 10: Every n-dim vector space over F is isomorphic to F^n. (noncanonical)
- Proof: V n-dimensional
  - Let $B=\{a_1,\ldots,a_n\}$ be a basis.
  - Define $T: V \to F^n$ by

$$\alpha = x_1\alpha_1 + \cdots + x_n\alpha_n \mapsto (x_1, \ldots, x_n) \in F^n$$

  - One-to-one
  - Onto

- Example: isomorphisms (F a subfield of R)

$$F^n = \{(x_1, \ldots, x_n) | x_i \in F\} \cong \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} | x_i \in F \right\}$$

$$P^n(F) = \{f : F \to F | f(x) = c_0 + c_1 x + \cdots + c_n x^n\} \cong F^{n+1}$$

Basis $\{1, x, x^2, \ldots, x^n\}$

$$c_0 + c_1 x + \cdots + c_n x^n \mapsto (c_0, c_1, \ldots, c_n)$$

There will be advantages in looking this way!